Opinion Dynamics over Influence Networks

Zahra Askarzadeh, Rui Fu, Abhishek Halder, Yongxin Chen, and Tryphon T. Georgiou

***Abstract*— The state of a societal group that is driven by social interactions amongst its members can be modeled as a probability vector, representing the relative strength of one's opinion within the group, that dictates the likely outcome of future interactions. Social interaction is then modeled by dynamics on the probability simplex, where the next state (probability vector) depends on the previous in a nonlinear manner that reflects interaction akin to McKean-Vlasov dynamics. In this short paper we outline theory that has been recently developed, and provide extensions, that rely on contractiveness of the state transition in the ℓ_1-norm. Our approach has links to other recent advances on monotone maps and differential positive systems.**

Keywords: Markov chains, opinion dynamics, reflected appraisal.

I. Overview

Models of social interaction has been the subject of a rapidly growing literature, see e.g. [1], [2], [3], [4], [5], [6], [7]. In the present note we overview recent developments [8] for a class of such models that have been motivated by the DeGroot-Friedkin dynamics, proposed and analyzed in [1], and we discuss generalizations.

The state of the pertinent system of agents is represented by a probability vector on the nodes (representing agents) that reflects the individual's self-confidence within the social structure, or their influence within the group. The discrete time index in the original model [1] represented issues being discussed. We consider several variants where the state is updated in real (running) time, without waiting for a consensus to be reached on issues. In doing so, we study maps

$$\boldsymbol{f} : \mathcal{S}_{n-1} \to \mathcal{S}_{n-1} : \boldsymbol{p} \mapsto \boldsymbol{f}(\boldsymbol{p}) \tag{1a}$$

that preserve the probability simplex (of column vectors $\boldsymbol{p}$)

$$\mathcal{S}_{n-1} := \{\boldsymbol{p} \in \mathbb{R}^n \mid p_i \geq 0, \sum_i p_i = 1\}.$$

Without loss of generality $\boldsymbol{f}(\boldsymbol{p})$ can be written in the form

$$\boldsymbol{f}(\boldsymbol{p}) = \boldsymbol{\Pi}(\boldsymbol{p})^T \boldsymbol{p} \tag{1b}$$

where $\boldsymbol{\Pi}(\boldsymbol{p})$ is a row stochastic matrix that depends (possibly nonlinearly) on the entries of $\boldsymbol{p}$, though, such a representation may not be unique in general. We focus on a particular structure of such maps where

$$\boldsymbol{\Pi}(\boldsymbol{p})^T = \boldsymbol{C}_0^T \boldsymbol{D}(\boldsymbol{p}) + \boldsymbol{C}_1^T (\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{p})), \tag{1c}$$

with $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ both row stochastic, and $\boldsymbol{D}(\boldsymbol{p})$ diagonal with entries in $[0,1]$; a typical special case being when $\boldsymbol{C}_0$ is the identity matrix.

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Note that $\boldsymbol{\Pi}(\boldsymbol{p})$ has nonnegative entries with rows summing to one for all $\boldsymbol{p} \in \mathcal{S}_{n-1}$. Depending on the values of the entries of $\boldsymbol{D}(\boldsymbol{p})$, the matrices $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ may model different aspects of interaction between agents (enhancing or diminishing influence of particular links between individuals).

We often specialize to the case where $\boldsymbol{C}_0$ is the identity matrix and $\boldsymbol{C}_1 = \boldsymbol{C} = [C]_{ij=1}^n$ encodes the influence of neighboring agents. Of particular interest are the exponentially-scaled transition kernel

$$\Pi_{ij}(x) = (1 - e^{-\gamma x_i})\delta_{ij} + e^{-\gamma x_i} C_{ij}, \text{ and its "opposite"} \tag{2a}$$

$$\overline{\Pi}_{ij}(x) = e^{-\gamma x_i}\delta_{ij} + (1 - e^{-\gamma x_i}) C_{ij}, \tag{2b}$$

as well as linearly-scaled kernels

$$\Pi_{ij}(x) = \gamma x_i \delta_{ij} + (1 - \gamma x_i) C_{ij}, \text{ and} \tag{3a}$$

$$\overline{\Pi}_{ij}(x) = (1 - \gamma x_i)\delta_{ij} + \gamma x_i C_{ij}, \tag{3b}$$

see [1]. Those two types of models provide rather insightful examples of the dynamics that one can expect from such social interactions.

In general, the nonlinear dynamics may display the complete spectrum of possible effects such as multiple equilibria, periodic orbits, and chaotic behavior. Models that capture such effects in social interactions originate in [9], [10], [11], [12], [13]. Among models proposed for opinion dynamics, the DeGroot model [9], the Friedkin-Johnson model [13], and the Krause model [12] have received the most attention. Several variations have also been proposed. For example, Xu etal. [14] introduced a Modified DeGroot-Friedkin model and analyzed it for the special case of doubly stochastic influence network, while Jia etal. [15] provided an analysis for the general case. Chen etal. in [2] proposed a continuous-time self-appraisal model. Tabatabaei etal. [16] considered possible effects of the group having "stubborn" individuals, and Ye etal. [17] considered dynamically changing network topologies.

The purpose of this article is to highlight an approach on stability analysis in [8] that relies on contractivity in ℓ_1 and provides easy-to-check sufficient conditions. In fact, the ℓ_1 distance represents a Finsler-Lyapunov function for stochastic maps [18], i.e., for maps preserving the simplex. This fact relates to structural monotonicity properties as brought out in [19], [20], [21]. Our approach differs from earlier literature which focuses mostly on continuous flows, although it relies on a similar conceptual setting (cf. [18], [19]). Following on in the present paper, we discuss continuous-time counterpart of the models as well as other extensions of the theory.

II. Contraction of ℓ_1-distances

It is standard and easy to show that if $\boldsymbol{\Pi}$ is a constant row stochastic matrix, the map $\boldsymbol{p} \mapsto \boldsymbol{\Pi}^T \boldsymbol{p}$ contracts in ℓ_1 (or, total variation). It is strictly contractive when all entries of $\boldsymbol{\Pi}$ are positive. Similar conclusions can be drawn for nonlinear maps of the particular form in (1). We discuss this next.

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Denote by $\mathcal{T}$ the tangent space of the probability simplex, i.e.,

$$\mathcal{T} := \{\boldsymbol{\delta} \in \mathbb{R}^n \mid \mathbb{1}^T \boldsymbol{\delta} = 0\}$$

with $\mathbb{1}$ a column vector of 1's. The Jacobian maps $\mathcal{T} \to \mathcal{T}$ and is

$$\mathrm{d}\boldsymbol{f} : (\delta_j)_{j=1}^n \mapsto \left(\sum_{i=1}^n \Pi_{ij}\delta_i\right)_{j=1}^n + \left(\sum_{i,k=1}^n \frac{\partial \Pi_{ij}}{\partial p_k} p_i \delta_k\right)_{j=1}^n$$

or, in a vectorial form

$$\mathrm{d}\boldsymbol{f} : \boldsymbol{\delta} \mapsto \left(\underbrace{\boldsymbol{\Pi}^T + \left[\frac{\partial \boldsymbol{\Pi}^T}{\partial p_1}\boldsymbol{p}, \ldots, \frac{\partial \boldsymbol{\Pi}^T}{\partial p_n}\boldsymbol{p}\right]}_{\boldsymbol{Q}^T}\right)\boldsymbol{\delta}. \tag{4}$$

Since $\mathbb{1}^T \boldsymbol{C}_i^T = \mathbb{1}^T$, for $i \in \{0, 1\}$, the columns on the second entry in the expression for $\boldsymbol{Q}^T$ satisfy

$$\mathbb{1}^T \left(\frac{\partial \boldsymbol{\Pi}^T}{\partial p_j}\boldsymbol{p}\right) = \mathbb{1}^T \left(\boldsymbol{C}_0^T \frac{\partial \boldsymbol{D}}{\partial p_j}\boldsymbol{p} - \boldsymbol{C}_1^T \frac{\partial \boldsymbol{D}}{\partial p_j}\boldsymbol{p}\right) = 0.$$

Hence,

$$\mathbb{1}^T \boldsymbol{Q}^T = \mathbb{1}^T \boldsymbol{\Pi}^T = \mathbb{1}^T. \tag{5}$$

The following key result is a consequence that, for the particular structure (1), the Jacobian satisfies (5) pointwise, i.e., for all $\boldsymbol{p} \in \mathcal{S}_{n-1}$. The idea is to consider the image of a path between two points $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$, and estimate the distance between their images under $\boldsymbol{f}$, showing that it contracts. The result is now given without proof below (for a proof see [8]).

Theorem 1: Let $\boldsymbol{f}(\cdot)$ be as in (1) with $\boldsymbol{D}(\boldsymbol{p})$ continuously differentiable, and suppose that the Jacobian matrix $\boldsymbol{Q}$ defined in (4) has strictly positive entries in $\mathcal{S}_{n-1}^o$. The following hold:

(a) $\boldsymbol{f}$ is strictly contractive in ℓ_1 in compact subsets of $\mathcal{S}_{n-1}^o$.

(b) Provided $\boldsymbol{f}$ has a fixed point in $\mathcal{S}_{n-1}^o$, this fixed point is the only fixed point and it is globally attracting.

For details we refer to the arXiv report [8].

Corollary 2: Let matrix $[\Pi_{ij}(\boldsymbol{p})]_{ij=1}^n$ be row-stochastic and differentiable in $\boldsymbol{p}$, and that $\boldsymbol{p}^\star$ is a fixed point of the map $\boldsymbol{f}$ in (1a), i.e., $\boldsymbol{p}^\star = \boldsymbol{\Pi}(\boldsymbol{p}^\star)\boldsymbol{p}^\star$. Suppose that the Jacobian of the map $\boldsymbol{f}$ evaluated at $\boldsymbol{p}^\star$ is such that, for a suitable integer m,

$$(\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^\star})^m$$

has strictly positive entries. Then $\boldsymbol{p}^\star$ is locally attractive.

Corollary 3: Let matrix $[\Pi_{ij}(\boldsymbol{p})]_{ij=1}^n$ be row-stochastic and differentiable in $\boldsymbol{p}$, and that $\boldsymbol{p}^i$, for $i = 0, 1, 2, \ldots, m-1$, is a periodic orbit for $\boldsymbol{f}$ in (1a), i.e., $\boldsymbol{p}^{(i+1)\mathrm{mod}(m)} = \boldsymbol{f}(\boldsymbol{p}^{(i)\mathrm{mod}(m)})$. Suppose that the product of the Jacobians

$$\left(\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^{(i+m)\mathrm{mod}(m)}}\right) \ldots \left(\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^{(i)\mathrm{mod}(m)}}\right)$$

has strictly positive entries for some i. Then, the periodic orbit is locally attractive.

A bound on the induced ℓ_1-incremental gain of stochastic maps in terms of the induced ℓ_1-gain of the Jacobian

$$\|\mathrm{d}\boldsymbol{f}|_\mathcal{T}\|_{(1)} := \max\{\|\boldsymbol{Q}^T\boldsymbol{\delta}\|_1 \mid \mathbb{1}^T\boldsymbol{\delta} = 0, \|\boldsymbol{\delta}\|_1 = 1\}$$

is given next. It strengthens the applicability of the approach by relaxing the positivity requirement on the Jacobian.

Proposition 4: Let $\boldsymbol{f}$ be a differentiable stochastic map as in (1a) and as before, the Jacobian $\mathrm{d}\boldsymbol{f}(\boldsymbol{p})|_\mathcal{T}$ is represented by a matrix $\boldsymbol{Q}(\boldsymbol{p})^T$. For any $\boldsymbol{p}^b, \boldsymbol{p}^a \in \mathcal{S}_{n-1}$,

$$\|\boldsymbol{f}(\boldsymbol{p}^b) - \boldsymbol{f}(\boldsymbol{p}^a)\|_1 \leq \max_{\boldsymbol{p} \in \mathcal{S}_{n-1}} \|\mathrm{d}\boldsymbol{f}(\boldsymbol{p})|_\mathcal{T}\|_{(1)} \|\boldsymbol{p}^b - \boldsymbol{p}^a\|_1,$$

and, in general,

$$\|\mathrm{d}\boldsymbol{f}|_\mathcal{T}\|_{(1)} = \frac{1}{2}\max_{j,k}\sum_{i=1}^n |(\boldsymbol{Q}(\boldsymbol{p}))_{ji} - (\boldsymbol{Q}(\boldsymbol{p}))_{ki}|. \tag{6}$$

The quantity (6) for the induced ℓ_1-norm of *linear maps* is the so-called Markov-Dobrushin coefficient of ergodicity [22], [23], [24], [25] that characterizes the contraction rate of Markov operators with respect to this norm (also, total variation). For nonlinear operators on probability simplices (nonlinear Markov Chains, see [26, Chapter 1]), the same is true. These propositions help provide certificates for stability of equilibria $\boldsymbol{p}^\star$ and highlight that the ℓ_1-distance is a Finsler-Lyapunov function in the sense of Forni and Sepulchre [18]. Thus, ℓ_1-contractivity of the dynamics $\boldsymbol{p}_{\text{next}} = \boldsymbol{f}(\boldsymbol{p})$, and stability of fixed points or periodic orbits, may be deduced from the infinitesimal properties of $\boldsymbol{f}$ in the ℓ_1-metric. The approach is illustrated in the next sections.

III. EXPONENTIAL-INFLUENCE MODELS

We analyze a few representative cases of (1) for $\boldsymbol{D}(\boldsymbol{p}) = \mathrm{diag}(r_1(\boldsymbol{p}), \ldots, r_n(\boldsymbol{p}))$ where $r_i(\boldsymbol{p})$ is either $1 - e^{-\gamma p_i}$ or $e^{-\gamma p_i}$, for some $\gamma > 0$. The first choice satisfies $r_i(0) = 0$ and $r_i'(0) = \gamma$, and thereby reinforces sites with relatively high values (entries of $\boldsymbol{p}$). The second choice has $r_i(0) = 1$ and $r_i'(0) = -\gamma$, has the tendency to do the opposite. Throughout we assume that $\boldsymbol{C}$ is an irreducible acyclic row-stochastic matrix, and we denote by $\boldsymbol{c}$ the unique (positive) Frobenius-Perron left eigenvector, i.e., $\boldsymbol{c}$ satisfies

$$\boldsymbol{C}^T\boldsymbol{c} = \boldsymbol{c},$$

and is normalized so that $\mathbb{1}^T\boldsymbol{c} = 1$. Because of the irreducibility assumption, $\boldsymbol{c}$ has positive entries.

Case $\mathbf{r(x) = 1 - e^{-\gamma x}}$ for $\gamma \leq 1$: We sum up the main conclusions in the following proposition.

Proposition 5: For any $\gamma \in [0, 1]$ consider

$$\boldsymbol{p}(k) \mapsto \boldsymbol{f}(\boldsymbol{p}(k)) =: \boldsymbol{p}(k+1), \text{ where} \tag{7a}$$

$$\boldsymbol{f}(\boldsymbol{p}(k)) = \left(\mathrm{diag}(\mathbb{1} - e^{-\gamma \boldsymbol{p}(k)}) + \boldsymbol{C}^T \mathrm{diag}(e^{-\gamma \boldsymbol{p}(k)})\right)\boldsymbol{p}(k). \tag{7b}$$

The map $\boldsymbol{f}$ is contractive in ℓ_1 and, starting from an arbitrary $\boldsymbol{p}(0) \in \mathcal{S}_{n-1}$, the limit $\boldsymbol{p}^\star = \lim_{k\to\infty}\boldsymbol{p}(k)$ exists, is unique, and its entries satisfy $e^{-\gamma p_i^\star} p_i^\star = \kappa c_i$, for some $\kappa > 0$.

The proof relies on the fact that the differential $\mathrm{d}\boldsymbol{f}^m(\cdot)$ is strictly contractive (cf. Corollary 2 of Theorem 1) and is detailed in our arXiv report [8].

Case $\mathbf{r(x) = 1 - e^{-\gamma x}}$ for $\gamma > 1$: This case is substantially different. Here, there can be several attractive points of equilibrium for the nonlinear dynamics in (7) and even more complicated nonlinear behavior. In fact, we believe that such a behavior may be more appropriate for models of opinion dynamics as it is reasonable to expect a different outcome depending on the starting point (that encapsulates confidence/beliefs of individuals). We illustrate the behavior with two numerical examples for 3-state dynamics to highlight differences with the case when $\gamma \leq 1$.

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1) Example: We consider (7) for a 3-state system (i.e., $n = 3$) with $\gamma = 4$ and

$$\boldsymbol{C} = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}. \tag{8}$$

The left Frobenius-Perron eigenvector of $\boldsymbol{C}$ is $(2/3, 1/6, 1/6)^T$. The fixed-point conditions for possible stationary distributions become

$$e^{-4p_1^\star} p_1^\star = \kappa \frac{2}{3},\ e^{-4p_2^\star} p_2^\star = \kappa \frac{1}{6},\ 2p_2^\star + p_1^\star = 1.$$

Upon eliminating κ between the first two, and substituting p_1 in terms of p_2, we obtain

$$\frac{1 - 2p_2^\star}{p_2^\star} e^{-4(1-3p_2^\star)} = 4. \tag{9}$$

This equation has the unique solution

$$\boldsymbol{p}^\star := (0.9904, 0.0048, 0.0048)^T.$$

It turns out that this is a locally attractive fixed point. This can be verified by evaluating the Jacobian of $\boldsymbol{f}$ at $\boldsymbol{p}^\star$ as

$$\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^\star} = \begin{bmatrix} 1.0113 & 0.3849 & 0.3849 \\ -0.0056 & 0.2303 & 0.3849 \\ -0.0056 & 0.3849 & 0.2303 \end{bmatrix}.$$

Even though the Jacobian has negative entries, it is still strictly contractive. Indeed, we explicitly evaluate the induced gain using Lemma 4 and this is

$$\|\mathrm{d}\boldsymbol{f}|_{\mathcal{T}}\|_{(1)} = \frac{1}{2} \max\{1.2528, 1.2528, 0.3092\} = 0.6264 < 1.$$

Thus $\boldsymbol{p}^\star$ is a stable fixed point. This analysis is consistent with simulations shown in Fig. 1. In the figure we depict trajectories (in different color) starting from random initial conditions that clearly tend to $\boldsymbol{p}^\star$.

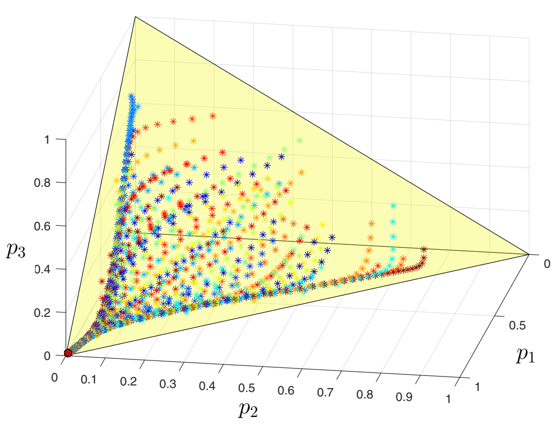

Fig. 1: Convergence of trajectories to a unique fixed point for the 3-state exponential model (7) with $\gamma = 4$ and influence matrix $\boldsymbol{C}$ given by (8).

2) Example: Once again we consider a 3-state system with $\gamma = 4$, but this time we take

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}. \tag{10}$$

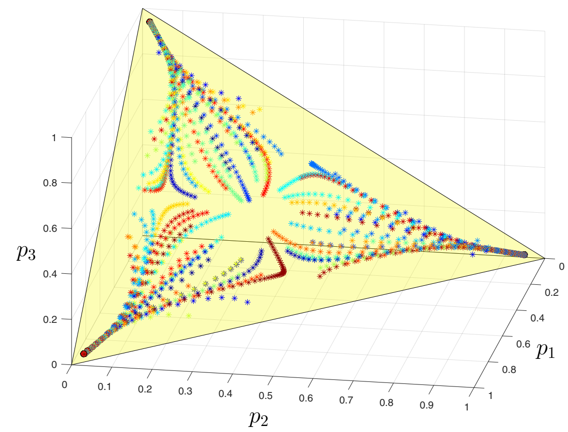

Fig. 2: For the 3-state exponential model (7) with $\gamma = 4$ and influence matrix $\boldsymbol{C}$ given by (10), trajectories converge to one of the three stable fixed points.

The fixed-point equations have 7 solutions (taking into account symmetries). Out of those, three are attractive fixed points with coordinates cyclically selected from $\{1 - a, a/2, a/2\}$ for $a = 0.046$. The remaining four are unstable fixed points. One is at the center $(1/3, 1/3, 1/3)^T$ (due to symmetry), and the rest have coordinates cyclically selected from $\{1 - a, a/2, a/2\}$ for $a = 0.874$. Just like the previous example, we can verify stability by computing the Jacobian $\mathrm{d}\boldsymbol{f}$ at fixed points. For instance, for the fixed point $\boldsymbol{p}_a^\star = (0.954, 0.023, 0.023)^T$, we have

$$\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}_a^\star} = \begin{bmatrix} 1.0620 & 0.4141 & 0.4141 \\ -0.0310 & 0.1718 & 0.4141 \\ -0.0310 & 0.4141 & 0.1718 \end{bmatrix},$$

and

$$\|\mathrm{d}\boldsymbol{f}|_{\mathcal{T}}\|_{(1)} = \frac{1}{2} \max\{1.2958, 1.2958, 0.4846\} = 0.6479 < 1.$$

Applying Lemma 4, we conclude that $\boldsymbol{p}_a^\star$ is a stable fixed point. For another fixed point $\boldsymbol{p}_b^\star = (0.1260, 0.4370, 0.4370)^T$, we have

$$\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}_b^\star} = \begin{bmatrix} 0.7004 & -0.0651 & -0.0651 \\ 0.1498 & 1.1302 & -0.0651 \\ 0.1498 & -0.0651 & 1.1302 \end{bmatrix},$$

and

$$\|\mathrm{d}\boldsymbol{f}|_{\mathcal{T}}\|_{(1)} = \frac{1}{2} \max\{1.9608, 1.9608, 2.3907\} = 1.1954 > 1.$$

Numerical evidence shown in Fig. 2 confirms that $\boldsymbol{p}_a^\star$ is stable and $\boldsymbol{p}_b^\star$ is unstable. Convergence of trajectories depends on the initial conditions with respect to the basins of attraction for the three stable fixed points. The qualitative behavior of the trajectories around the four unstable and three stable fixed points is illustrated in Fig. 3.

Case $\mathbf{r(x) = e^{-\gamma x}}$: When $\gamma \leq \mathbf{1}$ there is a unique fixed point and it is always globally attractive. The proof is detailed in our arXiv report [8]. When $\gamma > \mathbf{1}$, again there exists a unique fixed point in any dimension (any n) as well. However, in this case the nonlinear dynamics display diverse behaviors. Below we give three examples. In the first two the unique fixed point is attractive, but they differ, in that assurances for stability are drawn (for the second example) by computing the norm of the differential of higher iterants (2nd in

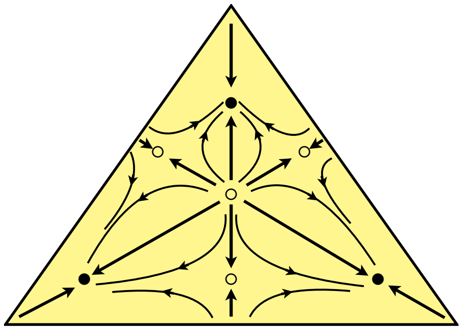

Fig. 3: The qualitative behavior of dynamics (7) with $\gamma > 1$ as observed in Fig. 2, where three stable fixed points (solid circles) and four unstable fixed points (empty circles) coexist on the simplex.

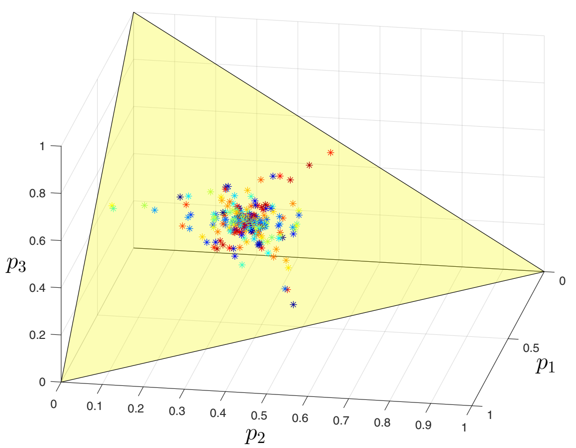

Fig. 4: For the 3-state exponential model with $\gamma = 4$ and influence matrix $\boldsymbol{C}$ given by (11), trajectories converge to the unique stable fixed point $\boldsymbol{p}^\star = (1/3, 1/3, 1/3)^T$.

this case). In the third example we observe a $2-$periodic attractive orbit.

3) Example: We consider a 3-state system with $\gamma = 4$, and

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}. \tag{11}$$

Since $\boldsymbol{C}$ is doubly stochastic, the unique fixed point is $\boldsymbol{p}^\star = (1/3, 1/3, 1/3)^T$, and we have

$$\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^\star} = \begin{bmatrix} -0.0880 & 0.5440 & 0.5440 \\ 0.5440 & -0.0880 & 0.5440 \\ 0.5440 & 0.5440 & -0.0880 \end{bmatrix},$$

and

$$\|\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^\star}\|_{(1)} = \frac{1}{2}\max\{1.2640, 1.2640, 1.2640\} = 0.6320 < 1.$$

Using Theorem 4, we conclude that $\boldsymbol{p}^\star$ is a stable fixed point.

4) Example: For $\gamma = 4$, now take

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$

The unique fixed point is again $\boldsymbol{p}^\star = (1/3, 1/3, 1/3)^T$. Here,

$$\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^\star} = \begin{bmatrix} -0.0880 & 0.5440 & 0.5440 \\ 0 & 0.4560 & 0.5440 \\ 1.0880 & 0 & -0.0880 \end{bmatrix},$$

and $\|\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^\star}\|_{(1)} = 1.1760$. However, $\|\mathrm{d}\boldsymbol{f}^2|_{\boldsymbol{p}^\star}\|_{(1)} = 0.7911$ which ensures local attractiveness.

5) Example: Once again we consider a 3-state system with $\gamma = 4$, but we now take

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0.8 & 0 & 0.2 \\ 0.8 & 0.2 & 0 \end{bmatrix}. \tag{12}$$

Uniqueness of a fixed point is guaranteed. This turns out to be

$$\boldsymbol{p}^\star = (0.4173, 0.1537, 0.4298)^T.$$

It turns out that

$$\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^\star} = \begin{bmatrix} -0.1261 & 0.6333 & 0.9031 \\ 0 & 0.2084 & 0.2258 \\ 1.1261 & 0.1583 & -0.1289 \end{bmatrix}$$

has ℓ_1-norm equal to 1.255, and so do the differentials of higher order iterants. However, a stable 2-periodic orbit now appears alternating between

$$\boldsymbol{p}^a = (0.1943, 0.1042, 0.7015)^T \text{ and } \boldsymbol{p}^b = (0.6450, 0.2005, 0.1545)^T.$$

The periodic orbit is locally attractive. The Jacobians at these two points are

$$\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^a} = \begin{bmatrix} 0.1024 & 0.4923 & 0.8873 \\ 0 & 0.3846 & 0.2218 \\ 0.8976 & 0.1231 & -0.1092 \end{bmatrix}$$

and

$$\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^b} = \begin{bmatrix} -0.1197 & 0.7290 & 0.6352 \\ 0 & 0.0888 & 0.1588 \\ 1.1197 & 0.1822 & 0.2060 \end{bmatrix},$$

respectively, and it can be verified that the norm of their product is $\|\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^a}\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^b}\|_{(1)} = 0.8750$. Interestingly, $\|\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^b}\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^a}\|_{(1)} = 0.7120$, which is different, but < 1 too (as expected). Stability can be ascertained by Corollary 3. An expalantion, as pointed out by an anonymous referee, is that as a particular state gets "more probable", it actually is associated with "less confidence", and hence there is indecision oscillating between alternatives.

Remark 6: We discuss the DeGroot-Friedkin Model and its Variants, where the nonlinear evolution corresponds to $r_i(\boldsymbol{p}) = \gamma p_i$ or $1 - \gamma p_i$, for $0 < \gamma \leq 1$, in the arXiv report [8] and provide analysis using the earlier theory.

IV. Extensions

We bring in two additional layers along which opinion models can be expanded. First, it is quite interesting to speculate about the effect of colluding sub-group in opinion forming. Indeed, everyday experience suggests that opinion is often reinforced within groups of like-minded individuals that draw confidence upon the collective wisdom, or lack of. Second, a continuous flow of confidence/influence in social interactions leads to nonlinear models of the McKean-Vlasov type. These two topics are touched next.

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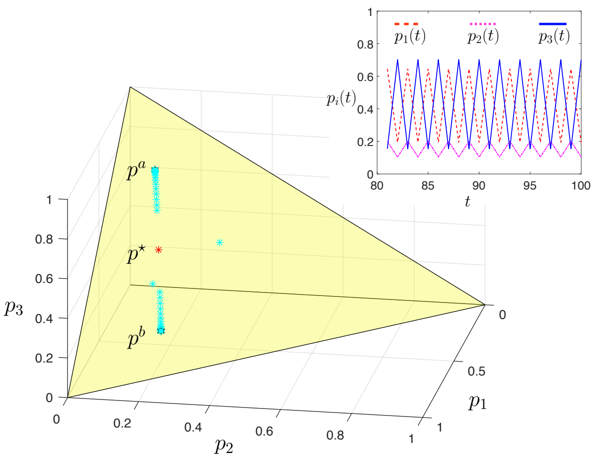

Fig. 5: For the 3-state exponential model with $\gamma = 4$, and $\boldsymbol{C}$ given by (12), the unique fixed point $\boldsymbol{p}^\star = (0.4173, 0.1537, 0.4298)^T$ is unstable and there is an attractive 2-periodic orbit between $\boldsymbol{p}^a$ and $\boldsymbol{p}^b$, verified by the time history (insert graph).

A. Collusion among subgroups

To account for such interactions, we use a stochastic matrix W to model the joint influence between group members by weighing their collective states via $\boldsymbol{r}(W\boldsymbol{p})$, which should be contrasted with individual-reinforcement of opinion/confidence modeled by $\boldsymbol{r}(\boldsymbol{p})$. This is independent and in addition to $\boldsymbol{C}$, which is used to model information flow over the total influence network. A reasonable choice for W is to be block diagonal where the blocks correspond to different subgroups of interacting individuals. The special case where W is identity matrix reduces to the earlier setting.

In fact, what we propose herein is an "interacting particle" analogue for nonlinear evolutions on the simplex, modeled as follows:

$$\begin{aligned}&\boldsymbol{p}(t+1) = \boldsymbol{\Pi}(\boldsymbol{p}(t))^T\boldsymbol{p}(t)\\ &= \Big(\operatorname{diag}(\boldsymbol{r}(W\boldsymbol{p}(t))) + \boldsymbol{C}^T(\boldsymbol{I} - \operatorname{diag}(\boldsymbol{r}(W\boldsymbol{p}(t))))\Big)\,\boldsymbol{p}(t). \quad (13)\end{aligned}$$

In particular, using a fixed-point argument as in [1] and Brouwer's fixed point theorem, we can establish existence results for the cases $\boldsymbol{r}(x) = x$ and $\boldsymbol{r}(x) = 1 - e^{-x}$, and a general stochastic matrix W.

Proposition 7: Let $\boldsymbol{r}(x) = x$ or $\boldsymbol{r}(x) = 1 - e^{-x}$, and W a stochastic matrix. Assume that $\boldsymbol{c}_k < \frac{1}{2}$ for all $k = 1, \ldots, n$. The nonlinear model (13), has at least one fixed point in the interior of $\mathcal{S}_{n-1}$.

We sketch the proof. Any fixed point of (13) must satisfy

$$\boldsymbol{p}_j = F_j(\boldsymbol{p}) := \frac{\boldsymbol{c}_j/(1-\boldsymbol{r}_j)}{\sum_k \boldsymbol{c}_k/(1-\boldsymbol{r}_k)}.$$

It is a bit cumbersome but straightforward to show that

$$F_j(\boldsymbol{p}) \le \frac{1}{1 + \sum_{k\neq j}\frac{\boldsymbol{c}_k}{\boldsymbol{c}_j(1-\boldsymbol{r}_k)}} < 1 - \epsilon.$$

On the other hand, starting from

$$\boldsymbol{p} \in \mathcal{S}_\epsilon := \{\boldsymbol{p} \in \mathcal{S}_{n-1} \mid \boldsymbol{p}_i \le 1 - \epsilon,\ \forall i = 1, \ldots, n\},$$

it is easy to see $\boldsymbol{r}(W\boldsymbol{p}) \in \mathcal{S}_\epsilon$ due to the facts that $\boldsymbol{r}(x) \le x$ and W is stochastic. Since $F(\mathcal{S}_\epsilon) \subset \mathcal{S}_\epsilon$ and F is continuous, by Brouwer's fixed-point theorem there exists $\boldsymbol{p}^*$ such that $\boldsymbol{p}^* = F(\boldsymbol{p}^*)$.

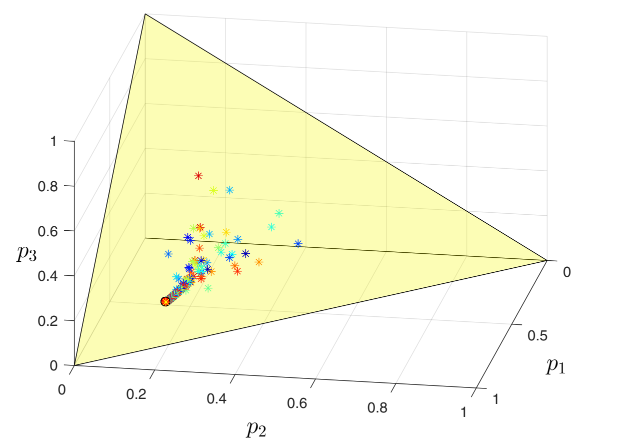

Fig. 6: For the 3-state model (13) with influence matrix $\boldsymbol{C}$ and W given by (14), trajectories converge to the unique fixed point $\boldsymbol{p}^\star = (0.6975, 0.1744, 0.1282)^T$.

The "nonlocal interaction" matrix W may in general introduce negative off-diagonal elements in Jacobian matrix. The theory in Section II applies on a case by case basis, but no general conclusion can be drawn at this point regarding global stability of particular class of models as we did earlier. Indeed, for $\boldsymbol{r}(x) = 1 - e^{-x}$, a matrix representation of the differential (4) becomes

$$\begin{aligned}\boldsymbol{Q}(\boldsymbol{p})^T \;\; = \;\; &\operatorname{diag}(\mathbb{1} - e^{-W\boldsymbol{p}}) + \boldsymbol{C}^T\operatorname{diag}(e^{-W\boldsymbol{p}})\\ &+(\boldsymbol{I} - \boldsymbol{C}^T)\operatorname{diag}(\boldsymbol{p}\circ e^{-W\boldsymbol{p}})W.\end{aligned}$$

This, in general, has negative entries, which however doesn't imply that the fixed point is unstable. The theory in Section II applies and attractiveness of equilibria can be ascertained by e.g., explicitly computing the ℓ_1-gain of $d\boldsymbol{f}|_{\mathcal{T}}$.

For a numerical example in $\mathcal{S}_2$, take $\boldsymbol{r}(x) = 1 - e^{-Wx}$,

$$\boldsymbol{C} = \begin{bmatrix}0.8 & 0.1 & 0.1\\ 0.4 & 0.2 & 0.4\\ 0.4 & 0.4 & 0.2\end{bmatrix}, \qquad W = \begin{bmatrix}0.5 & 0.5 & 0\\ 0.5 & 0.5 & 0\\ 0 & 0 & 1\end{bmatrix}. \quad (14)$$

Numerically (Fig. 6), we see that the system has a unique fixed point, $\boldsymbol{p}^\star = (0.6975, 0.1744, 0.1282)^T$, which is stable. This is consistent with element-wise positiveness of the Jacobian of (13) which is evaluated at $\boldsymbol{p}^\star$,

$$\mathrm{d}\boldsymbol{f}|_{\boldsymbol{p}^\star} = \begin{bmatrix}0.8932 & 0.2812 & 0.3068\\ 0.0872 & 0.5052 & 0.3068\\ 0.0196 & 0.2136 & 0.3865\end{bmatrix}.$$

It's worth mentioning that simulation with the same $\boldsymbol{C}$ but this time with $W = I_{3\times 3}$ gives $\boldsymbol{p}^\star = (0.8014, 0.0993, 0.0993)^T$. Hence, as expected, the influence between member of the sub-group has a strengthening effect.

B. Influence models over continuous time

The continuous space and time analogue of the nonlinear model (13) is the nonlinear Fokker-Plank-Vlasov equation

$$\rho_t = \Delta\rho + \nabla\cdot(\rho\nabla V(x)) + \nabla\cdot(\rho\nabla(\int W(x-y)\rho(y)dy)),$$

which has been used to model the evolution of densities for interacting particles systems under the influence of external potential V and interacting potential W [27], [28].

V. Concluding remarks

The theory allows drawing general conclusions on attractiveness of equilibria of nonlinear evolution models on probability simplices, i.e., stochastic evolutions. Besides the current interest in modeling dynamical interactions over social networks, the theory applies more broadly as similar models are pertinent in other types of interaction. Stability results as well as rates of convergence to equilibrium are important. Future research should also focus the effect of uncertainty and disturbances in such models.

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