The Convex Geometry of Integrator Reach Sets

Shadi Haddad and Abhishek Halder

Department of Applied Mathematics

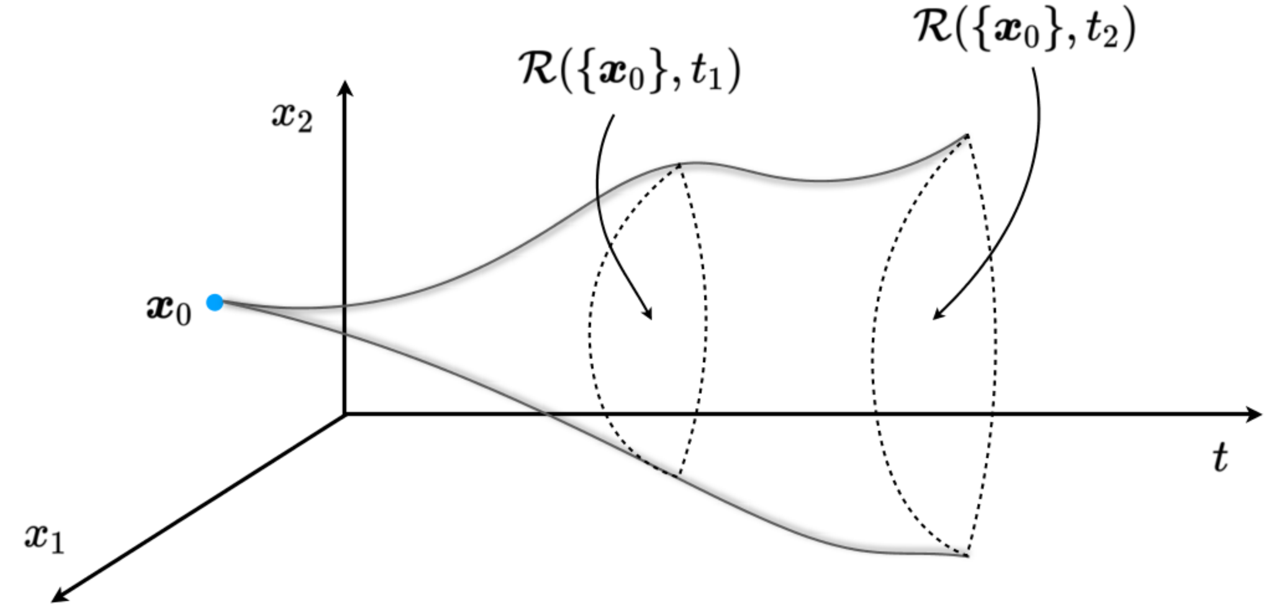
University of California, Santa Cruz

American Control Conference, July 3, 2020

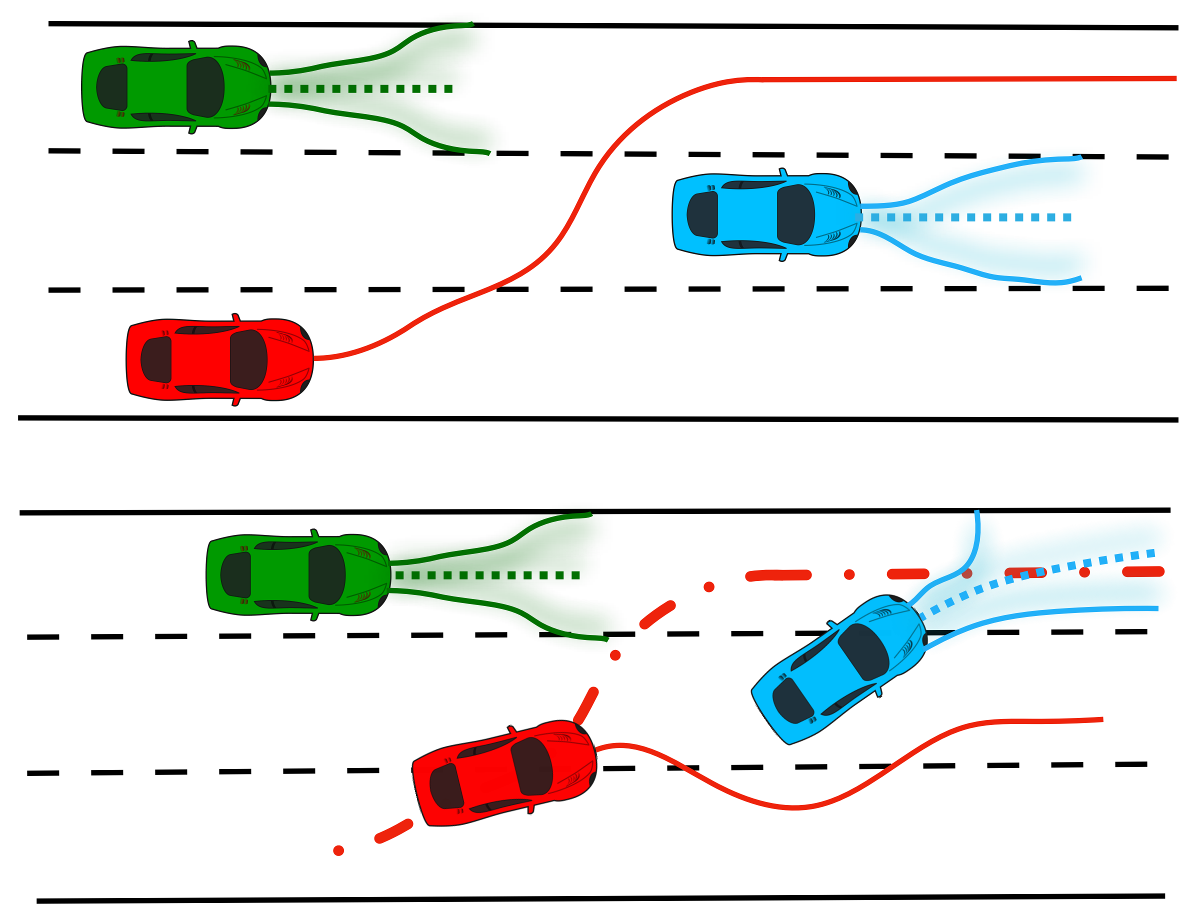
Support: Chancellor's Fellowship, UC Santa Cruz

Reach Sets

Predicting the states of an uncertain system



Safety critical applications such as motion planning & collision warning systems



Reach Set For Integrator Dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{x} \in \mathbb{R}^d, \quad u \in [-\mu, \mu]$$

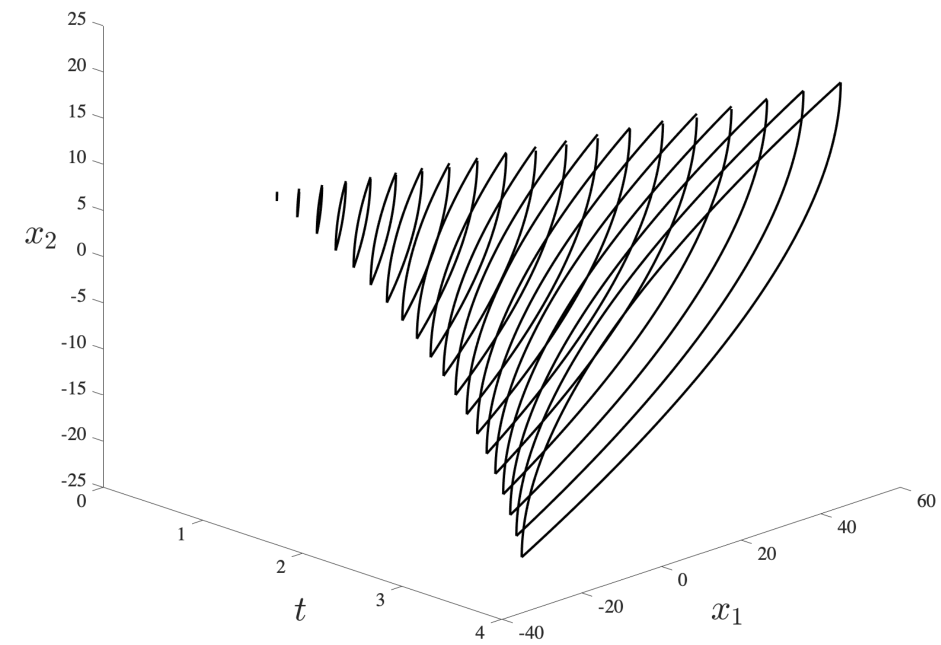
$$\mathbf{A} = [\mathbf{0} \quad \mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3 \quad \ldots \quad \mathbf{e}_{d-1}], \quad \mathbf{b} = \mathbf{e}_d$$

$$\mathcal{R}(\mathcal{X}_0, t) = \exp(t\mathbf{A})\mathcal{X}_o + \int_0^t \exp(s\mathbf{A})\mathbf{b}[-\mu, \mu]\mathrm{d}s$$

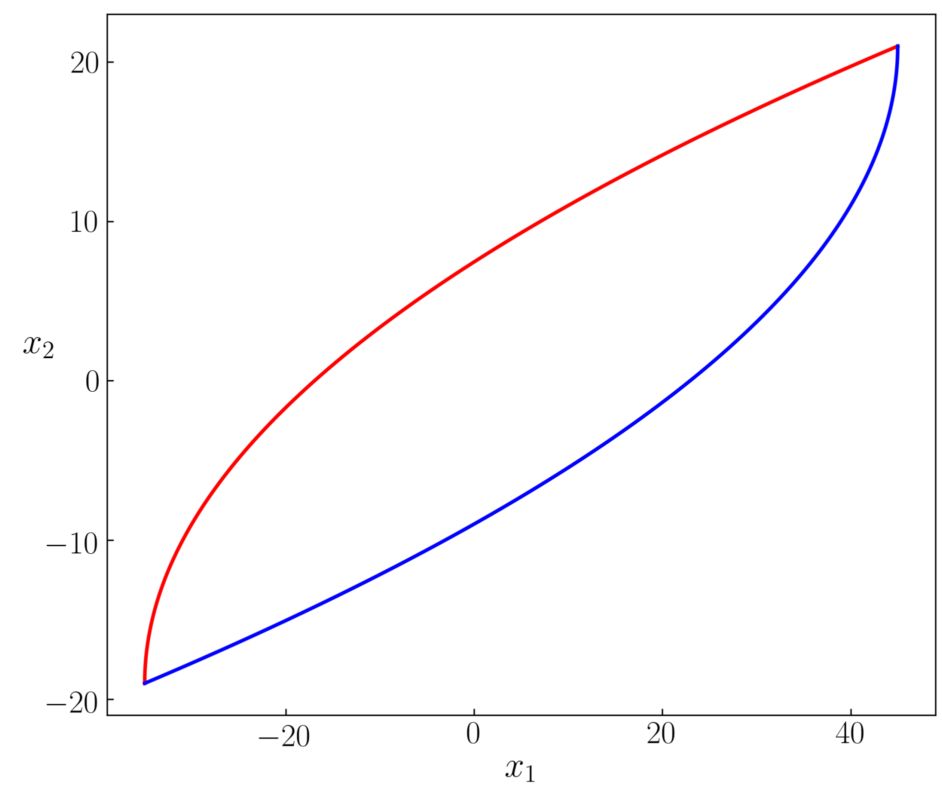
Minkowski sum

Nonlinear control systems of practical interest such as aerial and ground vehicles with bounded control: in normal form

Prototypical example in the systems-control literature on reach set computation



Reachable tube



Reach set

Integrator Reach Sets

Previous Studies	This Study
Approximation algorithms: elipsoidal, zonotopic, inner and outer approximation	Exact closed form formula for volume and diameter of the integrator reach sets
No quantitative assessment for comparison, content with statistical and graphical assessments	A foundation for benchmarking of algorithms

Support Function of the Integrator Reach Set

$$h_{\mathcal{R}(\mathcal{X}_0,t)}(\boldsymbol{y}) := \sup_{\boldsymbol{x}\in\mathcal{R}} \{\langle \boldsymbol{y}, \boldsymbol{x}\rangle \mid \boldsymbol{y} \in \mathbb{R}^d\}$$

$$= h_{\mathcal{X}_0}\!\left(\exp\left(t\boldsymbol{A}^\top\right)\boldsymbol{y}\right) + h_{\int_0^t \exp(s\boldsymbol{A})\boldsymbol{b}[-\mu,\mu]\mathrm{d}s}\left(\boldsymbol{y}\right)$$

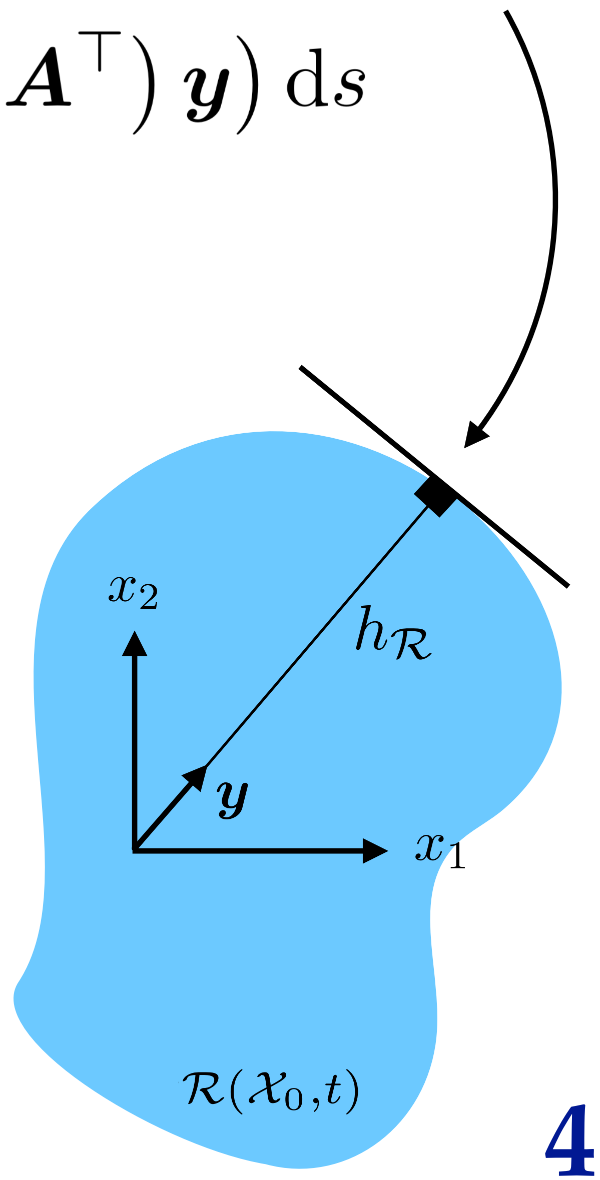
$$= h_{\mathcal{X}_0}\left(\exp\left(t\boldsymbol{A}^\top\right)\boldsymbol{y}\right) + \int_0^t h_{\boldsymbol{b}[-\mu,\mu]}\left(\exp\left(s\boldsymbol{A}^\top\right)\boldsymbol{y}\right)\mathrm{d}s$$

Supporting hyperplane

$$h_{\boldsymbol{b}[-\mu,\mu]}\left(\boldsymbol{y}\right) = \sup_{u\in[-\mu,\mu]} \langle \boldsymbol{y}, \boldsymbol{b}u\rangle = \mu|\langle \boldsymbol{y}, \boldsymbol{b}\rangle|$$

$$h_{\mathcal{R}(\mathcal{X}_0,t)}\left(\boldsymbol{y}\right) = \sup_{\boldsymbol{x}_0\in\mathcal{X}_0} \langle \boldsymbol{y}, \exp\left(t\boldsymbol{A}\right)\boldsymbol{x}_0\rangle + \mu\int_0^t |\langle \boldsymbol{y}, \boldsymbol{\xi}(s)\rangle|\,\mathrm{d}s$$

$$\boldsymbol{\xi}(s) := \begin{pmatrix} \frac{s^{d-1}}{(d-1)!} & \frac{s^{d-2}}{(d-2)!} & \dots & s & 1\end{pmatrix}^\top$$



Functional of the Integrator Reach Set

Volume:

Euclidean unit sphere

$$\mathrm{vol}\left(\mathcal{R}\left(\mathcal{X}_0, t\right)\right) = \frac{1}{d}\int_{\mathbb{S}^{d-1}} h_{\mathcal{R}(\mathcal{X}_0,t)}\left(\boldsymbol{\eta}\right)\, \mathrm{d}S_{\mathcal{R}(\mathcal{X}_0,t)}\left(\boldsymbol{\eta}\right), \quad \boldsymbol{\eta} \in \mathbb{S}^{d-1}$$

Lack analytical handle on the surface measure…

Alternative approach:

Minkowski sum of $n+1$ intervals

$$\mathrm{vol}\left(\mathcal{R}\left(\{\boldsymbol{x}_0\}, t\right)\right) = \mathrm{vol}\left(\int_0^t \exp\left(s\boldsymbol{A}\right)\boldsymbol{b}\left[-\mu, \mu\right]\mathrm{d}s\right)$$

$$= \mathrm{vol}\left(\lim_{n\to\infty}\sum_{i=0}^{n}\frac{t}{n}\exp\left(t_i\boldsymbol{A}\right)\boldsymbol{b}\left[-\mu, \mu\right]\right)$$

$$= \lim_{n\to\infty}\left(\frac{\mu t}{n}\right)^d \mathrm{vol}\left(\sum_{i=0}^{n}\exp\left(t_i\boldsymbol{A}\right)\boldsymbol{b}\left[-1, 1\right]\right)$$

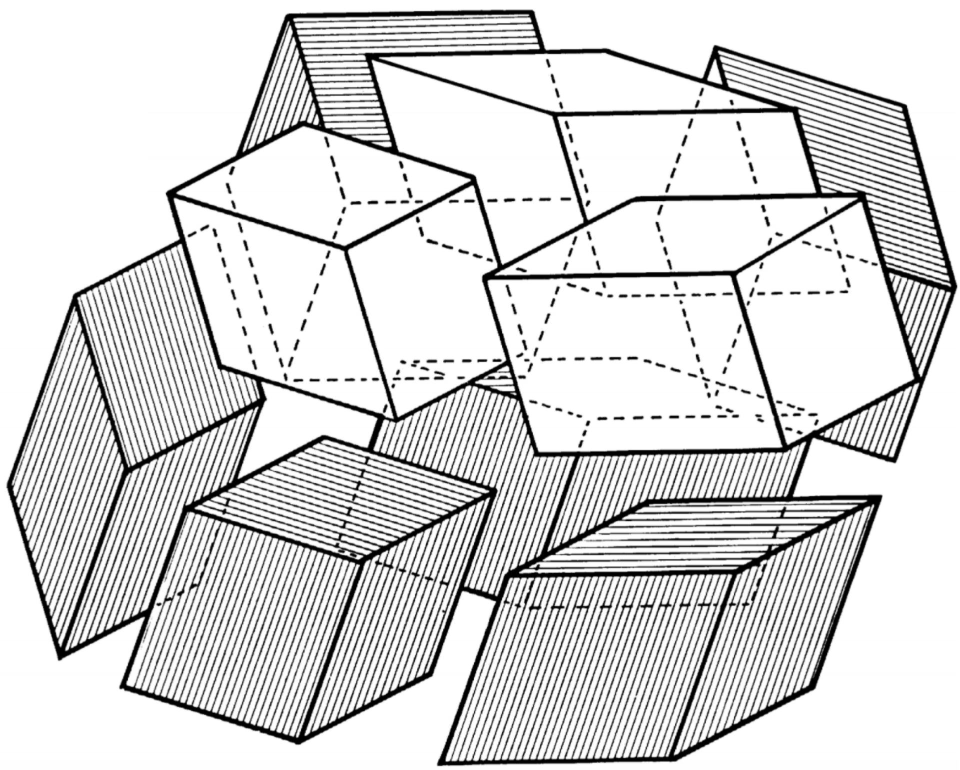
Functional of the Integrator Reach Set

Zonotope:

Generators

$$\mathcal{Z}_n := \left\{ \sum_{j=1}^{n} \gamma_j \boldsymbol{v}_j \,|\, \gamma_j \in [-1, 1], \boldsymbol{v}_j \in \mathbb{R}^d, j = 1, \ldots, n \right\}$$

$$h_{\mathcal{Z}_n}(\boldsymbol{y}) = \sum_{j=1}^{n} |\langle \boldsymbol{y}, \boldsymbol{v}_j \rangle|, \quad \boldsymbol{y} \in \mathbb{R}^d$$



Volume of zonotope:

$$\mathrm{vol}\,(\mathcal{Z}_n) = 2^d \sum_{1 \leq j_1 < j_2 < \ldots < j_d \leq n} |\det\,(\boldsymbol{v}_{j_1} | \boldsymbol{v}_{j_2} | \ldots | \boldsymbol{v}_{j_d})|$$

P. McMullen, "On zonotopes", Transactions of the American Mathematical Society, Vol. 159, 1971

Volume Formula

Theorem: Let $\boldsymbol{x}_0 \in \mathbb{R}^d$, $\mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$. Then:

$$\operatorname{vol}\left(\mathcal{R}\left(\{\boldsymbol{x}_0\}, t\right)\right) = \frac{(2\mu)^d t^{d(d+1)/2}}{\prod_{k=1}^{d-1} k!} \lim_{n\to\infty} \frac{1}{n^{d(d+1)/2}} \times \sum_{0 \le i_1 < i_2 < \ldots < i_d \le n} \prod_{1 \le \alpha < \beta \le d} (i_\beta - i_\alpha)$$

Further Simplification of the Volume Formula

$$\mathrm{vol}(\mathcal{R}(\{\boldsymbol{x}_0\}, t)) = (2\mu)^d t^{\frac{d(d+1)}{2}} \prod_{k=1}^{d-1} \frac{k!}{(2k+1)!}$$

Proof sketch:

Step 1: The following sum returns a polynomial in n of degree $d(d+1)/2$.

$$\sum_{0 \le i_1 < i_2 < \ldots < i_d \le n} \quad \prod_{1 \le p < q \le d} (i_q - i_p)$$

Step 2: By Euler-Maclaurin formula, the leading coefficient $c(d)$ of this polynomial:

$$c(d) = \int_{x_1=0}^{x_1=1} \int_{x_2=0}^{x_2=x_1} \cdots \int_{x_d=0}^{x_d=x_{d-1}} (x_1 - x_2) \ \ldots (x_{d-1} - x_d)\, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_d$$

$$= \sum_{\sigma \in S_d} \mathrm{sgn}(\sigma) \frac{1}{\prod_{i=1}^{d} (\sigma_1 + \sigma_2 + \ldots + \sigma_i)},$$

where $\mathrm{sgn}(\sigma) := (-1)^m$

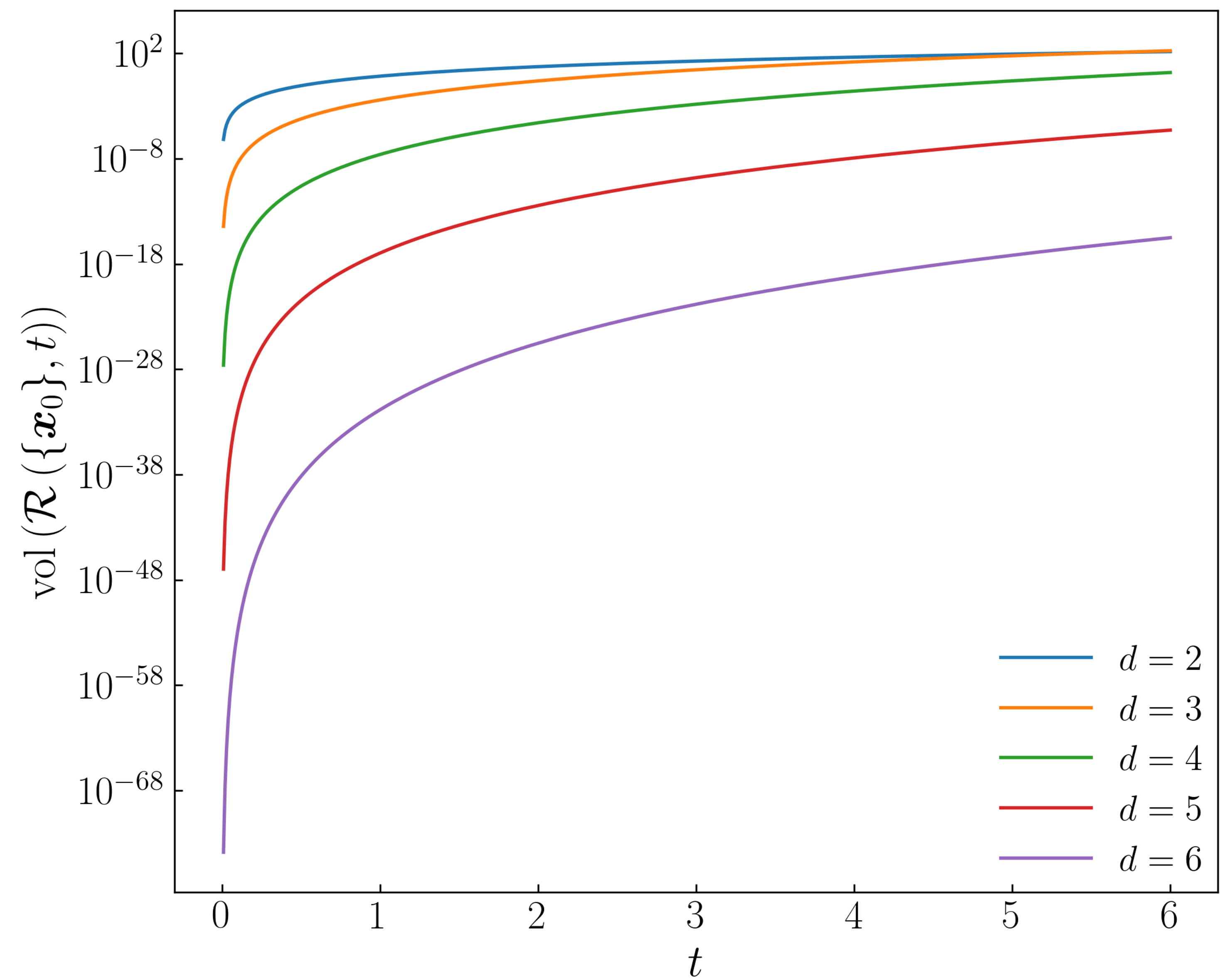
$m := \{\#(i,j) \mid i < j, \sigma(i) > \sigma(j)\}$

Step 3: Using Pfaffians: $\quad c(d) = \prod_{k=1}^{d-1} \frac{(k!)^2}{(2k+1)!}$

Scaling Law for Volume

Volume of integrator reach set vs time

$$\boldsymbol{x}_0 \in \mathbb{R}^d,\ \mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$$



Volume of integrator reach set vs time

Functional of the Integrator Reach Set

Width:

Direction of width

$$w_{\mathcal{R}(\mathcal{X}_0,t)}(\boldsymbol{\eta}) := h_{\mathcal{R}(\mathcal{X}_0,t)}(\boldsymbol{\eta}) + h_{\mathcal{R}(\mathcal{X}_0,t)}(-\boldsymbol{\eta})$$

R. Schneider, "Convex Bodies", 2014

$$= 2\mu \int_0^t |\langle \boldsymbol{\eta}, \boldsymbol{\xi}(s)\rangle| \, \mathrm{d}s$$

Diameter:

$$\mathrm{diam}\left(\mathcal{R}\left(\mathcal{X}_0, t\right)\right) := \max_{\boldsymbol{\eta}\in\mathbb{S}^{d-1}} w_{\mathcal{R}(\mathcal{X}_0,t)}(\boldsymbol{\eta})$$

for $\boldsymbol{x}_0 \in \mathbb{R}^d$, $\mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$

Diameter Formula

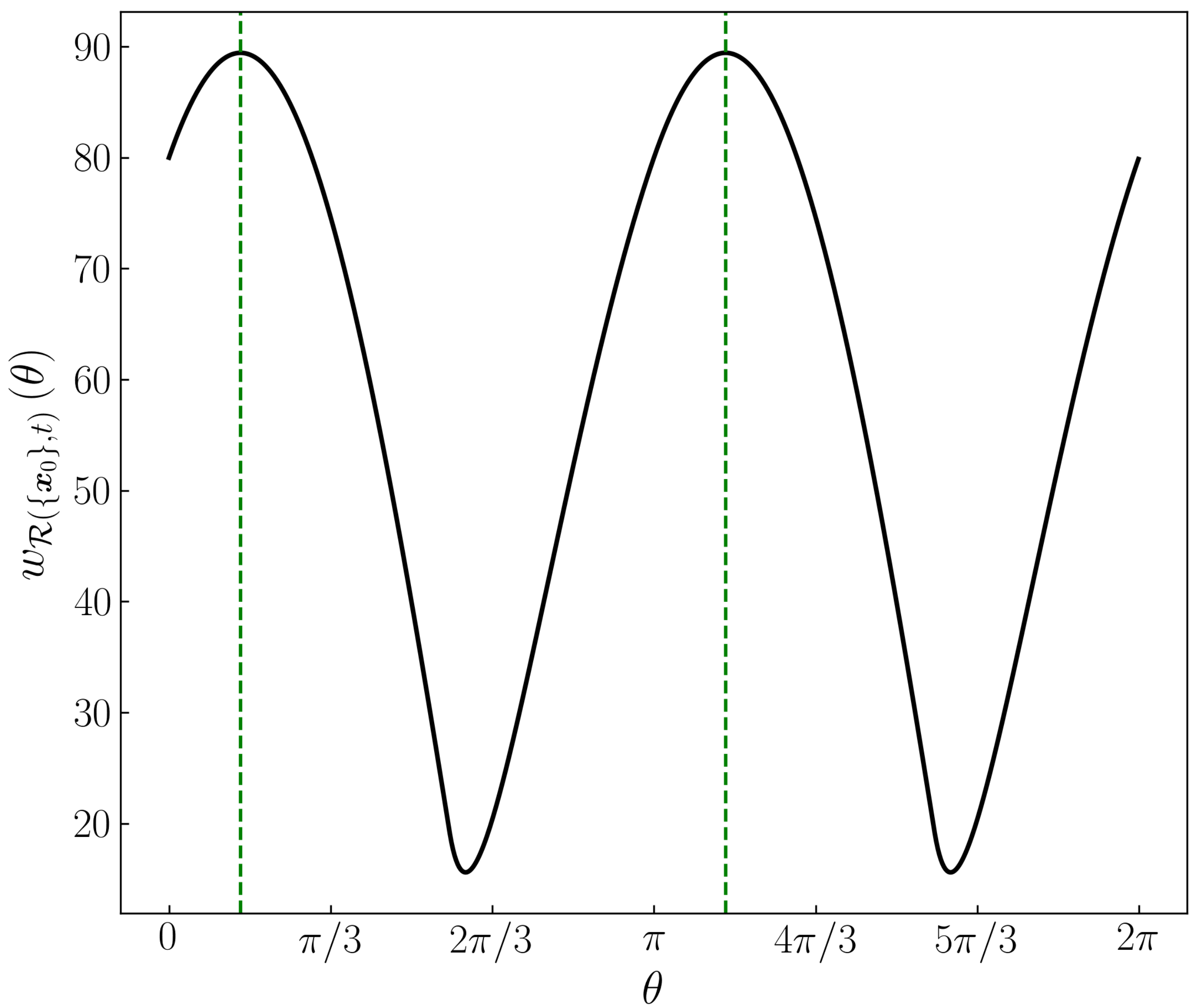
Theorem: Let $\boldsymbol{x}_0 \in \mathbb{R}^d$, $\mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$**. Then:**

$$\operatorname{diam}\left(\mathcal{R}\left(\{\boldsymbol{x}_0\}, t\right)\right) = 2\mu \parallel \boldsymbol{\zeta}(t) \parallel_2 = 2\mu \left\{ \sum_{j=1}^{d} \left(\frac{t^j}{j!} \right)^2 \right\}^{\frac{1}{2}}$$

2 dimensional case:

$$\boldsymbol{\eta} \equiv (\cos\theta, \sin\theta)^\top,\ \theta \in \mathbb{S}^1$$

$$\operatorname{diam}\left(\mathcal{R}\left(\{\boldsymbol{x}_0\}, t\right)\right) = \mu t \sqrt{t^2 + 4}$$

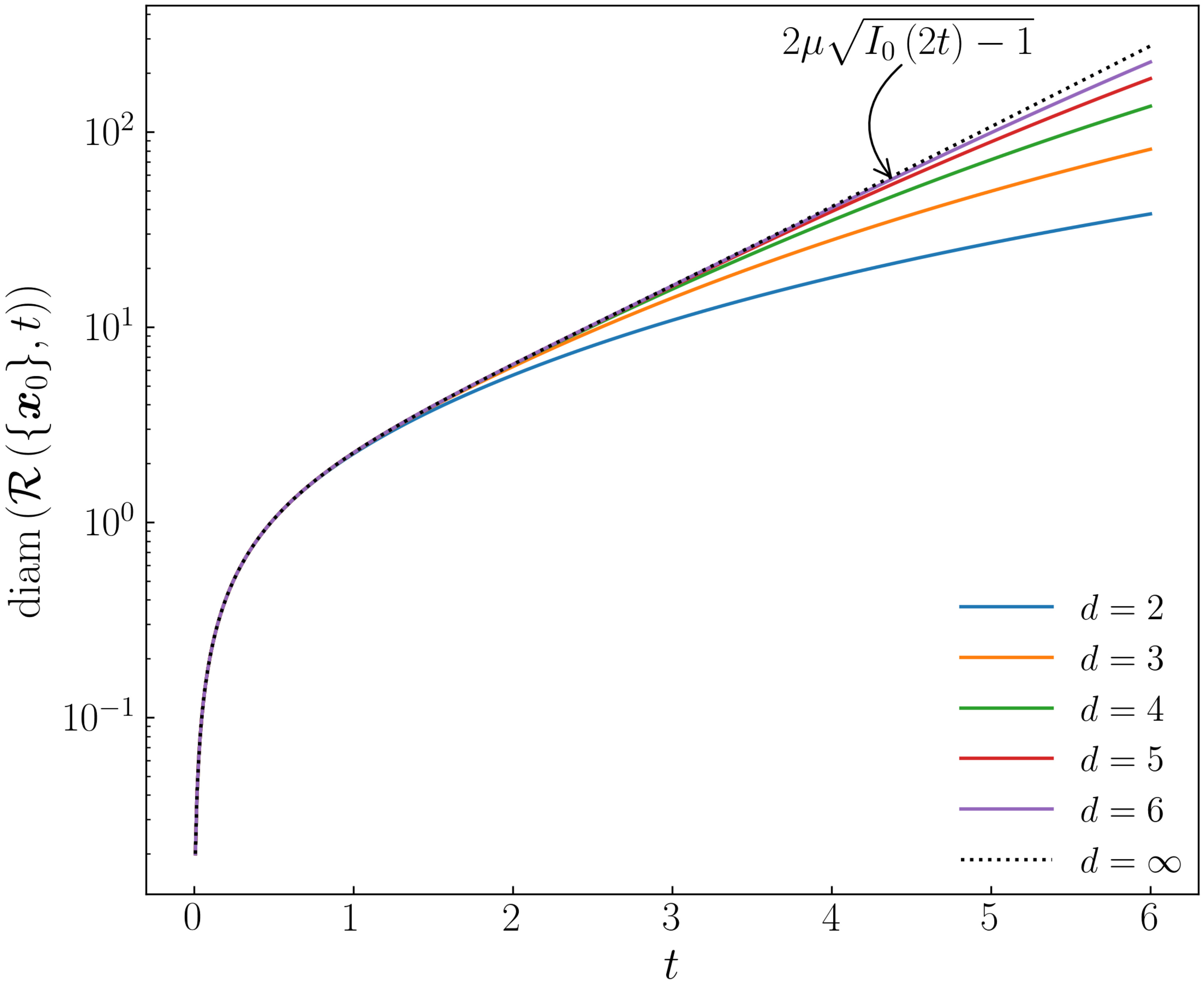


Width the of 2D integrator reach set vs θ

Scaling Law for Diameter

Diameter the of integrator reach set vs time

$\boldsymbol{x}_0 \in \mathbb{R}^d,\ \mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$



Diameter of integrator reach set vs time

Summary

Volume formula:

$$\text{vol}\left(\mathcal{R}\left(\mathcal{X}_0, t\right)\right) = \left(2\mu\right)^d t^{\frac{d(d+1)}{2}} \prod_{k=1}^{d-1} \frac{k!}{(2k+1)!}$$

Diameter formula:

$$\text{diam}\left(\mathcal{R}\left(\{\boldsymbol{x}_0\}, t\right)\right) = 2\mu \parallel \boldsymbol{\zeta}(t) \parallel_2 = 2\mu \left\{ \sum_{j=1}^{d} \left(\frac{t^j}{j!}\right)^2 \right\}^{\frac{1}{2}}$$

Future work

Multi-input case

Algorithms for computing reach sets of state feedback linearizable systems

Thank You