**AERO 320: Numerical Methods**
Fall 2013

Final Examination

**Academic Integrity**. I will enforce the Aggie Code of Honor: "An Aggie does not lie, cheat or steal, or tolerate those who do." There is a zero tolerance policy for academic dishonesty.

Name:________________________________________ Date:____________________

For this exam you only need a pen/pencil and a calculator. Write your answers in the spaces provided. If you need more space, work on the other side of the page.
Unless explicitly mentioned, use your calculator's precision for all your calculations.

1. Consider $f(x) = \ln x$. We want to compute $f'(x)$ at $x = 3$, using different numerical methods. $(3 + 5 + 5 + 7 = 20 \text{ points})$

   This problem is same as Homework 7, Problem 2.

   (a) Find the exact expression for $\dfrac{d}{dx}\ln x$. Evaluate your answer numerically at $x = 3$ to find $f'_{\text{exact}}(3)$. Keep up to 8 significant digits.

   $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$. Hence, $f'_{\text{exact}}(3) = \dfrac{1}{3} \approx 0.33333333$ (up to 8 significant digits).

   (b) Compute the *second order central difference* approximation $f'_{h_1}(3)$ with step size $h_1 = 0.4$. Keep up to 8 significant digits.

   $f'_{h_1}(3) = \dfrac{\ln(3+0.4) - \ln(3-0.4)}{2 \times 0.4} = \dfrac{\ln(3.4) - \ln(2.6)}{0.8} \approx 0.33532998$ (up to 8 significant digits).

   (c) Compute the *second order central difference* approximation $f'_{h_2}(3)$ with step size $h_2 = \dfrac{h_1}{2} = 0.2$. Keep up to 8 significant digits.

   $f'_{h_2}(3) = \dfrac{\ln(3+0.2) - \ln(3-0.2)}{2 \times 0.2} = \dfrac{\ln(3.2) - \ln(2.8)}{0.4} \approx 0.33382848$ (up to 8 significant digits).

   (d) Using your answers in part (b) and (c), compute the *second order Richardson extrapolation* $f'_{\text{Richardson}}(3)$. Keep up to 8 significant digits.

   $f'_{\text{Richardson}}(3) = f'_{h_2}(3) + \dfrac{1}{2^2 - 1}\left(f'_{h_2}(3) - f'_{h_1}(3)\right) \approx 0.33332798$ (up to 8 significant digits).

2. Consider numerically computing the integral $\int_0^{\frac{\pi}{2}} \cos x \, dx$. $(5+5+5+(2+3)=20$ points)

This problem is same as Lab 18.

(a) Compute the integral using *midpoint method* with the $x$-axis partition $\left[0, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

$$I_a = \left(\frac{\pi}{4} - 0\right) \cos\frac{\pi}{8} + \left(\frac{\pi}{2} - \frac{\pi}{4}\right) \cos\frac{3\pi}{8} = \frac{\pi}{4}\left(\cos\frac{\pi}{8} + \cos\frac{3\pi}{8}\right) \approx 1.0262.$$

(b) Compute the integral using *trapezoid method* with the same partition as in part (a).

$$I_b = \frac{1}{2} \times \left(\frac{\pi}{4} - 0\right) \times \left(\cos 0 + \cos\frac{\pi}{4}\right) + \frac{1}{2} \times \left(\frac{\pi}{2} - \frac{\pi}{4}\right) \times \left(\cos\frac{\pi}{4} + \cos\frac{\pi}{2}\right) = \frac{\pi}{8}\left(1 + \sqrt{2}\right) \approx 0.9481.$$

(c) Compute the integral using *three point Simpson's method* with the same partition as in part (a).

$$I_c = \frac{1}{6}\left(\frac{\pi}{2} - 0\right)\left(\cos 0 + 4\cos\frac{\pi}{4} + \cos\frac{\pi}{2}\right) = \frac{\pi}{12}\left(1 + \frac{4}{\sqrt{2}} + 0\right) = \frac{\pi}{12}\left(1 + 2\sqrt{2}\right) \approx 1.0023.$$

(d) Compute the *exact value* of the integral. Compute the *relative errors* in your answers in part (a), (b) and (c).

Exact value: $\int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_{x=0}^{x=\frac{\pi}{2}} = 1$.

Keeping five significant digits in the answers in (a), (b) and (c), the relative error in (a): $|1 - 1.0262| = 0.0262$, relative error in (b): $|1 - 0.9481| = 0.0519$, relative error in (c): $|1 - 1.0023| = 0.0023$.

3. $(10 + 15 = 25$ points)

(a) Use Taylor series method to solve the ordinary differential equation

$$\frac{dy}{dx} = x + y - xy, \qquad y(0) = 1,$$

at $x = 0.5$, to compute $y_{\text{taylor}}(0.5)$. Keep up to terms with $x^4$, i.e. neglect terms with $x^5$ or higher powers.

Ch.6 in textbook, Problem 1 (sent as practice problem)

Since the initial condition $x_0 = 0$, hence we expand $y(x)$ in Taylor series about $x = 0$, as shown below.

$$y_{\text{taylor}}(x) = y(0) + y'(0)(x-0) + \frac{y''(0)}{2!}(x-0)^2 + \frac{y'''(0)}{3!}(x-0)^3 + \frac{y''''(0)}{4!}(x-0)^4.$$

We know that $y(0) = 1$. Hence $y'(0) = 0 + y(0) - 0 \times y(0) = 1$,
$y''(0) = 1 + y'(0) - 0 \times y'(0) - y(0) = 1 + 1 - 0 - 1 = 1$,

$y'''(0) = y''(0) - 2y'(0) - 0 \times y''(0) = 1 - 2 - 0 = -1$,
$y''''(0) = y'''(0) - 3y''(0) - 0 \times y'''(0) = -1 - 3 - 0 = -4$. Substituting these back in the Taylor series, we get

$$y_{\text{taylor}}(x) = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{6} \Rightarrow y_{\text{taylor}}(0.5) = 1.59375.$$

(b) Repeat part (a) with Euler method to compute $y_{\text{euler}}(0.5)$ with step size $h = 0.1$.

Ch.6 in textbook, Problem 5 (sent as practice problem)

In the Euler method, we have $y_{\text{euler}}(x+h) = y_{\text{euler}}(x) + hy'(x)$. Hence
$y(0.1) = y(0+0.1) = y(0) + 0.1 \times y'(0) = 1 + 0.1 \times 1 = 1.1$,
$y(0.2) = y(0.1+0.1) = y(0.1) + 0.1 \times y'(0.1) = 1.1 + 0.1 \times (0.1 + 1.1 - 0.1 \times 1.1) = 1.209$,
$y(0.3) = y(0.2+0.1) = y(0.2) + 0.1 \times y'(0.2) =$
$1.209 + 0.1 \times (0.2 + 1.209 - 0.2 \times 1.209) = 1.32572$,
$y(0.4) = y(0.3+0.1) = y(0.3) + 0.1 \times y'(0.3) =$
$1.32572 + 0.1 \times (0.3 + 1.32572 - 0.3 \times 1.32572) = 1.4485204$,
$y(0.5) = y(0.4+0.1) = y(0.4) + 0.1 \times y'(0.4) =$
$1.4485204 + 0.1 \times (0.4 + 1.4485204 - 0.4 \times 1.4485204) = 1.575431624$.
Thus, $y_{\text{euler}}(0.5) = 1.575431624$.

4. $(10 + (4+4+2) + (1+4) = 25$ points)

(a) Determine the values of $a$ and $b$ such that the function $S(x)$ given below, is a linear spline.

$$S(x) = \begin{cases} x, & -1 \le x \le 0.5, \\ 0.5 + 2(x-a), & 0.5 \le x \le 2, \\ x + b, & 2 \le x \le 4. \end{cases}$$

Similar to the solution of Homework 6, Problem 2(a).

From continuity, $S(0.5) = 0.5 = 0.5 + 2(0.5 - a) \Rightarrow a = 0.5$. Similarly,
$S(2) = 0.5 + 2(2 - 0.5) = 2 + b \Rightarrow b = 1.5$.

(b) Use Newton's method to derive an algorithm to compute the square root of a real number $N$. For $N = -1$, use your algorithm to perform two Newton iterations (find $x_1$ and $x_2$) with $x_0 = \frac{1}{\sqrt{3}}$. For this initial guess, does the algorithm converge or diverge?

Similar to Homework 2, Problem 3 (also Problem 14, p. 67 in textbook).

This is same as solving the nonlinear equation $x^2 - N = 0$, which yields
$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$.

For $N = -1$, our algorithm becomes $x_{n+1} = \frac{1}{2}\left(x_n - \frac{1}{x_n}\right)$.

For $x_0 = \frac{1}{\sqrt{3}}$, we get $x_1 = \frac{1}{2}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = -\frac{1}{\sqrt{3}}$, $x_2 = \frac{1}{2}\left(-\frac{1}{\sqrt{3}} + \sqrt{3}\right) = \frac{1}{\sqrt{3}}$.

The algorithm neither converges, nor diverges. It oscillates between $\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$.

(c) Consider the following system of linear equations:

$$\begin{array}{ccccccc} x_1 & + & 4x_2 & + & \alpha x_3 & = & 6, \\ 2x_1 & - & x_2 & + & 2\alpha x_3 & = & 3, \\ \alpha x_1 & + & 3x_2 & + & x_3 & = & 5. \end{array}$$

How many solutions can be obtained for the vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ when $\alpha \neq \pm 1$? Give reasons in support of your answer.

This problem is same as Homework 3, Problem 2(a).

Exactly one unique solution, because $\det \begin{pmatrix} 1 & 4 & \alpha \\ 2 & -1 & 2\alpha \\ \alpha & 3 & 1 \end{pmatrix} = \alpha^2 - 1 \neq 0$, for all $\alpha \neq \pm 1$.

5. For the following *multiple-choice questions*, only one out of the three options is correct. Clearly mark the option that you think is correct. You DO NOT need to write any reasons in support of your answer. ($5 \times 2 = 10$ points)

(a) Consider the *orders of convergence* $\alpha_{\text{secant}}$ and $\alpha_{\text{newton}}$ for Secant method and Newton's method, respectively, to compute the root of a nonlinear equation. We can say

i. always $1 < \alpha_{\text{secant}} < 2$, and $\alpha_{\text{newton}} = 2$.

ii. always $\alpha_{\text{secant}} = 1$, and $\alpha_{\text{newton}} = 2$.

iii. if the root appears only once, then $1 < \alpha_{\text{secant}} < 2$, and $\alpha_{\text{newton}} = 2$; otherwise $\alpha_{\text{secant}} = \alpha_{\text{newton}} = 1$.

iii.

(b) Consider four different algorithms you learnt for *polynomial interpolation*: monomial basis (solving a system of linear equations with Vandermonde matrix), Lagrange polynomial, Neville's algorithm, and divided differences. For a fixed set of data points

i. all four algorithms produce the same polynomial.

ii. only Neville's algorithm and divided differences produce the same polynomial.

iii. four algorithms produce four different polynomials.

i.

(c) Consider solving a system of linear equations $Ax = b$ using *Jacobi* and *Gauss-Seidel* algorithms. For $A = \begin{pmatrix} 3 & -5 & 2 \\ 5 & 4 & 3 \\ 2 & 5 & 3 \end{pmatrix}$

i. both Jacobi and Gauss-Seidel method will diverge.
ii. both Jacobi and Gauss-Seidel method will converge.
iii. no conclusions can be made about the convergence, just by looking at matrix $A$.

iii.

(d) *LU decomposition* for a square matrix

i. always exists.
ii. if exists, then may not be unique.
iii. if exists, then is unique.

ii.

(e) For any $n \times n$ orthogonal matrix $Q$, its *condition number* with respect to 2-norm is denoted as $\kappa_2(Q)$, and with respect to Frobenius norm is denoted as $\kappa_F(Q)$. Then

i. $\kappa_2(Q) = 1$, $\kappa_F(Q) = \sqrt{n}$.
ii. $\kappa_2(Q) = 1$, $\kappa_F(Q) = n$.
iii. $\kappa_2(Q) = \kappa_F(Q) = 1$.

ii.

# Some useful information

Theorems and definitions

- *Second order central difference approximation* with step size $h$, for $f'(x)$, is

$$f_h'(x) = \frac{f(x+h) - f(x-h)}{2h} + O\left(h^2\right).$$

- $n^{\text{th}}$ order *Richardson extrapolation* for approximating $f'(x)$, assuming $h_2 < h_1$, is

$$f'_{\text{Richardson}} = f'_{h_2} + \frac{1}{2^n - 1}\left(f'_{h_2} - f'_{h_1}\right).$$

- Area of a rectangle = width $\times$ height.

- Area of a trapezoid $= \frac{1}{2} \times$ distance between the parallel sides $\times$ sum of the lengths of the parallel sides.

- To approximate $\int_a^b f(x)\ dx$, *three point Simpson's method* fits a parabola through three points: $(a, f(a))$, $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$, and $(b, f(b))$. Then the integral is approximated as the area under this parabola:

$$\int_a^b f(x)\ dx \approx \frac{(b-a)}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

- The Taylor series of $y$ as a function of $x$, around the point $x = x_0$, is given by

$$y_{\text{taylor}}(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \frac{y''''(x_0)}{4!}(x - x_0)^4 + \ldots$$

- The Euler method for solving an ODE, with a given step size $h$, is

$$y_{\text{euler}}(x+h) = y_{\text{euler}}(x) + hy'(x).$$

- *Newton's method* to compute roots of a nonlinear equation $f(x) = 0$, is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

- Definition of *order of convergence* $\alpha$, for a sequence of approximations $y_n$ to the real value $y$:

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^{\alpha}} = \lambda,$$

  where $e_n = |y_n - y|$. The number $\lambda$ is called *asymptotic error constant*. For large $n$:

$$\alpha \approx \frac{\log(e_{n+1}) - \log(e_n)}{\log(e_n) - \log(e_{n-1})}.$$

# Some useful information

Theorems and definitions (contd.)

- A matrix $A$ with elements $a_{ij}$, $i = 1, \ldots, m$, and $j = 1, \ldots, n$, is called *diagonally dominant* if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \qquad \text{for each} \quad i = 1, 2, \ldots, m.$$

If $A$ is diagonally dominant, then both *Jacobi* and *Gauss-Seidel* algorithms for solving a system of linear equations of the form $Ax = b$, converge for any initial guess.

- An $n \times n$ orthogonal matrix $Q$ is defined as

$$QQ^\top = Q^\top Q = I,$$

where $I$ denotes the $n \times n$ identity matrix.

- Norms of an $m \times n$ matrix $A$:

$$\begin{aligned} \text{Frobenius norm: } \| A \|_F &= \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} = \sqrt{\operatorname{tr}(AA^\top)} \\ \text{2-norm: } \| A \|_2 &= \sqrt{\lambda_{\max}(AA^\top)} \end{aligned}$$

- Condition numbers of an invertible matrix $A$:

$$\begin{aligned} \text{with respect to Frobenius norm: } \kappa_F(A) &= \| A \|_F \, \| A^{-1} \|_F \\ \text{with respect to 2 norm: } \kappa_2(A) &= \| A \|_2 \, \| A^{-1} \|_2 \end{aligned}$$