AERO 320: Numerical Methods
Fall 2013

Mid Term 1

Academic Integrity. I will enforce the Aggie Code of Honor: "An Aggie does not lie, cheat or steal, or tolerate those who do." There is a zero tolerance policy for academic dishonesty.

Name:____________________________________ Date:________________

For this exam you only need a pen/pencil and a calculator. Write your answers in the spaces provided. If you need more space, work on the other side of the page.
Unless explicitly mentioned, use your calculator's precision for all your calculations.

1. The function $\sin(x)$ has a Taylor expansion of the form

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - ...$$

(a) For $x = 1$, approximate $\sin(x)$ using the right-hand-side with (i) just the first term, (ii) the first and second terms, and (iii) the three terms.

(b) Compute the exact value of $\sin(x)$ at $x = 1$ (radians). Estimate *absolute* and *relative errors* as well as *significant digits* for the three approximations above.

(c) What kind of error is this?

(d) How can you reduce it?

(e) Just as we did for other sequences in class, we can determine the *order of convergence* α for this sequence. What is it?

2. Consider the fixed point iteration

$$x_{k+1} = (\alpha + 1)\, x_k - x_k^2, \quad k = 0, 1, 2, ...$$

where α satisfying $1 \geq \alpha \geq 1/2$ is given.

(a) Show that the iteration converges for any initial guess x_0 satisfying

$$\alpha/2 \leq x_0 \leq \alpha/2 + 1.$$

(b) What would happen if I started from $x_0 = \alpha/3$?

3. For a quadratically convergent sequence, we know that

$$e_n = \lambda e_{n-1}^2,$$

where $e_n = |x_n - x_{\text{exact}}|$. Show that

$$\lambda e_n = (\lambda e_0)^{2^n}.$$

4. It is possible for Newton's method to iterate forever. To see this notice that the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

cycles back and forth around a point a if

$$x_{n+1} - a = -(x_n - a).$$

(a) Show that for this to happen, the function $f(x)$ must satisfy

$$\frac{f'(x)}{f(x)} = \frac{1}{2(x-a)}.$$

(b) From your answer to part (a), prove that the function must be

$$f(x) = \operatorname{sign}(x-a)\ \sqrt{|x-a|}.$$

(c) By hand, draw a plot of the function $f(x)$ found in part (b), for $a = 2$ and $x \in [0, 4]$. Using your plot, explain why Newton's method will fail to converge to the root $x = a$.

5. Consider the following:

```
float u = 1.;
for (int i=1; 1 + u > 1; i++){
u = u/2.;
cout << u << i << endl;
}
```

   (a) Is this an infinite loop? If not, what can you say about the final value of u? Explain.

   (b) Will your answer change if I define u as double precision? Explain.

6. (a) Determine the *minimum number of iterations* n, required by the *bisection method* to converge to within an absolute error tolerance of ε, starting from the initial interval (a, b).

(b) Starting with the initial guess $x_0 = 0.3$, perform two iterations (i.e. find x_1, x_2) of *Newton's method*, to solve the equation $f(x) = \dfrac{1}{x} - 3 = 0$. What is the exact root for this problem?

(c) Using the last two iterations in part (b), and the convergence properties you know about Newton's method, estimate the *asymptotic error constant* λ.

Some useful information

<u>Theorems and definitions</u>

- Definition of rate of convergence for a sequence of approximations y_n to the real value y:

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^\alpha} = \lambda,$$

  where $e_n = |y_n - y|$. For large n:

$$\alpha \approx \frac{\log(e_{n+1}) - \log(e_n)}{\log(e_n) - \log(e_{n-1})}.$$

- Theorem for bisection: The error at the n^{th} iteration is bounded according to

$$|x_n - x| \leq \frac{b-a}{2^n},$$

  where (a, b) is the initial interval.

- The function $\operatorname{sign}(\cdot)$ is defined as follows:

$$\operatorname{sign}(y) = \begin{cases} +1 & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -1 & \text{if } y < 0. \end{cases}$$

<u>Some iterative methods</u>

- $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (order of convergence: $\alpha = 2$ for single root, $\alpha = 1$ for multiple roots)
- $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$ (order of convergence: $1 < \alpha < 2$)

<u>C++</u>

- A non-zero number divided by zero returns "Inf".
- Zero divided by zero returns "NaN". So does the logarithm of a negative number.