AERO 320: Numerical Methods
Fall 2013

Mid Term 2

Academic Integrity. I will enforce the Aggie Code of Honor: "An Aggie does not lie, cheat or steal, or tolerate those who do." There is a zero tolerance policy for academic dishonesty.

Name:____ Date:____________________

For this exam you only need a pen/pencil and a calculator. Write your answers in the spaces provided. If you need more space, work on the other side of the page.
Unless explicitly mentioned, use your calculator's precision for all your calculations.

1. Let $A = \begin{pmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix}.$ (15 + 5 + 15 = 35 points)

   (a) Perform LU decomposition for matrix A. Show all the calculations in *exact arithmetic* (i.e. use fractions throughout).

   See solution for Homework 4, Problem 1(a).

   (b) Use your answer in part (a) to compute $\det(A)$.

   See solution for Homework 4, Problem 1(b).

   (c) Solve $Ax = b$ using the LU decomposition.

   See solution for Homework 4, Problem 1(d).

2. Consider two vectors $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad y = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$ (4 + 5 + 16 + 5 + 5 = 35 points)

   (a) Compute the vector norms $\| x \|_2$, and $\| y \|_2$.

   $\| x \|_2 = \sqrt{1^2 + 2^2} = \sqrt{5}, \qquad \| y \|_2 = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$

   (b) Compute the matrix $A = x\, y^\top$.

   $$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}.$$

   (c) Using your answer in part (b), compute the norms $\| A \|_F$, and $\| A \|_2$.

   $$\| A \|_F = \sqrt{\sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}^2} = \sqrt{3^2 + 4^2 + 6^2 + 8^2} = \sqrt{125}.$$

   $$AA^\top = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 25 & 50 \\ 50 & 100 \end{pmatrix}, \Rightarrow \lambda\left(AA^\top\right) = 0,\ 125,\ \Rightarrow \| A \|_2 = \sqrt{\lambda_{\max}\left(AA^\top\right)} = \sqrt{125}.$$

In this case, $\| A \|_F = \| A \|_2$.

(d) For any 2×1 vector b, can the system $Az = b$ have unique solution? Why/why not?

No, because $\det(A) = (3 \times 8) - (6 \times 4) = 0$.

(e) If we change the 2×1 vectors x and y, will your answer to part (d) change? Why/why not?

No, there will not be a unique solution even if the vectors x and y are changed. To see this, take any 2×1 vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then $A = x\, y^\top = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$, and hence $\det(A) = x_1 x_2 y_1 y_2 - x_1 x_2 y_1 y_2 = 0$, for any 2×1 vectors x and y.

3. Suppose we have recorded following data from some experiment: $(15 + 5 + 10 = 30$ points)

$$(x_0, y_0) = (1, 1); \qquad (x_1, y_1) = (2, 8); \qquad (x_2, y_2) = (3, 27).$$

(a) Compute the Lagrange interpolating polynomial that passes through these data points.

Since we have 3 data points, the Lagrange interpolating polynomial is

$$\begin{aligned} y = \sum_{i=0}^{2} y_i\, \ell_i(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}\, y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}\, y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\, y_2 \\ &= \frac{(x-2)(x-3)}{(1-2)(1-3)} \times 1 + \frac{(x-1)(x-3)}{(2-1)(2-3)} \times 8 + \frac{(x-1)(x-2)}{(3-1)(3-2)} \times 27 \\ &= 6x^2 - 11x + 6. \end{aligned}$$

(b) From your answer in part (a), predict the value of y at $x = 2.5$. Also, compute the *absolute* and *relative error* in your prediction, provided the *true* value of y at $x = 2.5$ is 15.625.

$y_{\text{predict}} = 6 \times (2.5)^2 - 11 \times 2.5 + 6 = 16.$

Absolute error $= |y_{\text{true}} - y_{\text{predict}}| = |15.625 - 16| = 0.375.$

Relative error $= \dfrac{|y_{\text{true}} - y_{\text{predict}}|}{|y_{\text{true}}|} = \dfrac{0.375}{15.625} = 0.024.$

(c) Instead of Lagrange polynomial, if we interpolate using *any* polynomial of the form $y = c_0 + c_1 x + c_2 x^2$, then we must find the coefficient vector $c = \{c_0 \quad c_1 \quad c_2\}^\top$. From our experimental data, can we *uniquely* find the vector c? Why/why not?

Yes, we can.

The vector c must satisfy

$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 27 \end{pmatrix}$. This system of linear equations admits *unique* solution vector c because $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = 2 \neq 0$.

Some useful information

<u>Theorems and definitions</u>

- To perform LU decomposition means to express the matrix A as $A = LU$, where L and U are *lower* and *upper triangular matrices*, respectively. A linear system $Ax = b$ is then solved as follows. First solve for y in $Ly = b$, and then solve for x in $Ux = y$.

- Properties of trace and determinant:

$$\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)\,, \qquad \det\,(AB) = \det\,(A)\;\det\,(B)\,.$$

- Norms of an $n \times 1$ vector x:

$$\|\, x \,\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|\, x \,\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}, \qquad \|\, x \,\|_\infty = \max_{i=1,\ldots,n} |x_i|.$$

- Norms of an $m \times n$ matrix A:

$$\begin{aligned}
\text{Frobenius norm: } \|\, A \,\|_F &= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{\mathrm{tr}(AA^\top)} \\
\text{2-norm: } \|\, A \,\|_2 &= \sqrt{\lambda_{\max}\,(AA^\top)} \\
\text{1-norm: } \|\, A \,\|_1 &= \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}| \\
\infty\text{-norm: } \|\, A \,\|_\infty &= \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|
\end{aligned}$$

- Lagrange polynomials: For $n+1$ points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n), the Lagrange interpolating polynoimal is $y(x) = \sum_{i=0}^{n} y_i\, \ell_i\,(x)$, where $\ell_i\,(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \dfrac{(x - x_j)}{(x_i - x_j)}$.