Finite Horizon LQG Density Regulator with Wasserstein Terminal Cost

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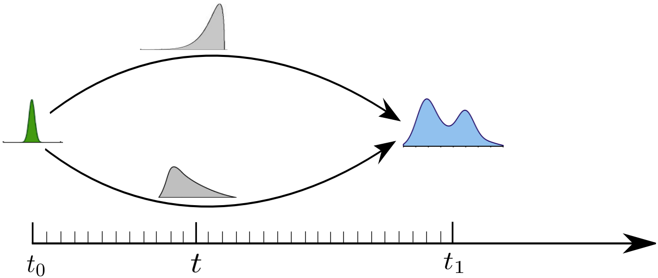
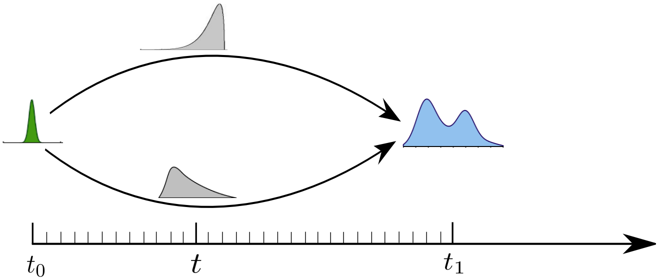
Finite Horizon Density Regulator: Outlook



Finite Horizon Density Regulator: Outlook



AMS/IP Studies in Advanced Mathematics, vol 39, 2007, pp 23-35.

Optimal control of the Liouville equation

R. W. Brockett

Notes on the Control of the Liouville Equation

Roger Brockett

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Optimal Steering of a Linear Stochastic System to a Final Probability Distribution, Part I

Yongxin Chen, *Student Member, IEEE*, Tryphon T. Georgiou, *Fellow, IEEE*, and Michele Pavon

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Optimal Steering of a Linear Stochastic System to a Final Probability Distribution, Part II

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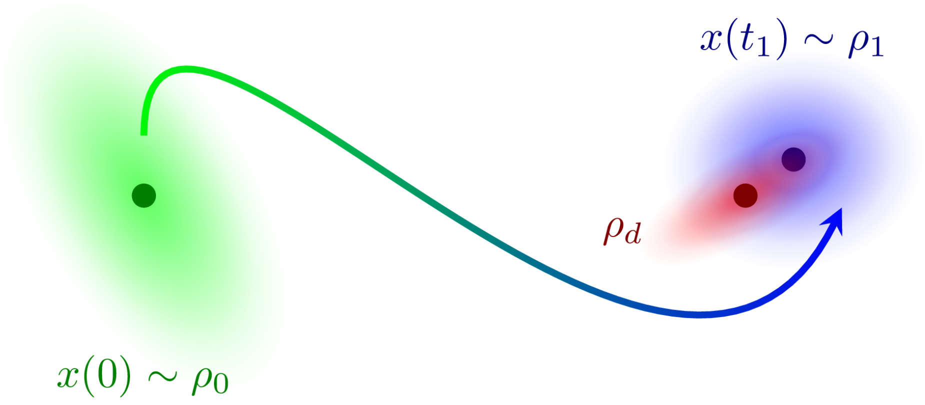
Density Regulator with Terminal Cost



LQG State Regulator

$$\min_{u \in \mathcal{U}} \phi\left(x_1, x_d\right) + \mathbb{E}_x \left[\int_0^{t_1} \left(x^\top Q x + u^\top R u\right) \mathrm{d}t \right]$$

$$\mathrm{d}x(t) = Ax(t)\,\mathrm{d}t + Bu(t)\,\mathrm{d}t + F\,\mathrm{d}w(t),$$

$$x(0) = x_0 \text{ given}, \quad x_d \text{ given}, \quad t_1 \text{ fixed},$$

Typical terminal cost: MSE

$$\phi\left(x_1, x_d\right) = \mathbb{E}_{x_1}\left[(x_1 - x_d)^\top M (x_1 - x_d)\right]$$

LQG Density Regulator

$$\min_{u \in \mathcal{U}} \varphi\left(\rho_1, \rho_d\right) \; + \; \mathbb{E}_x\left[\int_0^{t_1} (x^\top Q x \; + \; u^\top R u)\, \mathrm{d}t\right]$$

$$\mathrm{d}x(t) = Ax(t)\,\mathrm{d}t + Bu(t)\,\mathrm{d}t + F\,\mathrm{d}w(t),$$

$$x(0) \sim \rho_0 \text{ given}, \;\; x_d \sim \rho_d \text{ given}, \;\; t_1 \text{ fixed},$$

Proposed terminal cost: MMSE

$$\varphi\left(x_1, x_d\right) = \inf_{y \sim \rho \in \mathcal{P}_2(\rho_1, \rho_d)} \mathbb{E}_y\left[(x_1 - x_d)^\top M (x_1 - x_d)\right],$$

where $y := (x_1, x_d)^\top$

Problem Statement: LQG Density Regulator

$$\varphi(\rho_1, \rho_d)$$

$$\min_{u \in \mathcal{U}} \inf_{y \sim \rho \in \mathcal{P}_2(\rho_1, \rho_d)} \mathbb{E}_y \left[(x_1 - x_d)^\top M (x_1 - x_d) \right] + \mathbb{E}_x \left[\int_0^{t_1} (x^\top Q x + u^\top R u) \, \mathrm{d}t \right]$$

$$\mathrm{d}x(t) = Ax(t)\,\mathrm{d}t + Bu(t)\,\mathrm{d}t + F\,\mathrm{d}w(t),$$

$$x(0) \sim \rho_0 = \mathcal{N}(\mu_0, S_0), \quad x_d \sim \rho_d = \mathcal{N}(\mu_d, S_d),$$

$$t_1 \text{ fixed}, \quad \mathcal{U} = \{u \,:\, u(x,t) = K(t)x + v(t)\}$$

However, $\varphi\left(\mathcal{N}\left(\mu_1, S_1\right), \mathcal{N}\left(\mu_d, S_d\right)\right)$ equals

$$\left(\mu_1 - \mu_d\right)^\top M \left(\mu_1 - \mu_d\right) +$$

$$\min_{C \in \mathbb{R}^{n\times n}} \operatorname{tr}\left((S_1 + S_d - 2C)M\right) \text{ s.t. } \begin{bmatrix} S_1 & C \\ C^\top & S_d \end{bmatrix} \succeq 0$$

However, $\varphi\left(\mathcal{N}\left(\mu_1, S_1\right), \mathcal{N}\left(\mu_d, S_d\right)\right)$ equals

$$\left(\mu_1-\mu_d\right)^\top M\left(\mu_1-\mu_d\right)+$$

$$\min_{C\in\mathbb{R}^{n\times n}} \operatorname{tr}\left(\left(S_1+S_d-2C\right)M\right) \ \text{s.t.} \ \begin{bmatrix} S_1 & C \\ C^\top & S_d \end{bmatrix} \succeq 0$$

$$\Updownarrow$$

$$\max_{C\in\mathbb{R}^{n\times n}} \operatorname{tr}\left(CM\right) \quad \text{s.t.} \quad CS_d^{-1}C^\top \succeq 0$$

$$\Updownarrow$$

$$C^* = S_1 S_d^{\frac{1}{2}}\left(S_d^{\frac{1}{2}} S_1 S_d^{\frac{1}{2}}\right)^{-\frac{1}{2}} S_d^{\frac{1}{2}}$$

This gives

$$\varphi\left(\mathcal{N}\left(\mu_1, S_1\right), \mathcal{N}\left(\mu_d, S_d\right)\right) = \left(\mu_1 - \mu_d\right)^\top M \left(\mu_1 - \mu_d\right)$$

$$+\mathrm{tr}\left(MS_1 + MS_d - 2\left[\left(\sqrt{S_d}MS_1\sqrt{S_d}\right)\left(\sqrt{S_d}S_1\sqrt{S_d}\right)^{-\frac{1}{2}}\right]\right)$$

Applying maximum principle:

$$K^o(t) = R^{-1}B^\top P(t),$$

$$v^o(t) = R^{-1}B^\top\left(z(t) - P(t)\mu(t)\right)$$

∞ dim. TPBVP $\rightsquigarrow 2\left(n + \frac{n(n+1)}{2}\right)$ dim. TPBVP

$$\begin{pmatrix} \dot{\mu}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} A & BR^{-1}B^\top \\ Q & -A^\top \end{pmatrix} \begin{pmatrix} \mu(t) \\ z(t) \end{pmatrix},$$

$$\dot{S}(t) = (A + BK^o)S(t) + S(t)(A + BK^o)^\top + FF^\top,$$

$$\dot{P}(t) = -A^\top P(t) - P(t)A - P(t)BR^{-1}B^\top P(t) + Q,$$

Boundary conditions:

$$\mu(0) = \mu_0, \quad z(t_1) = M(\mu_d - \mu_1),$$

$$S(0) = S_0, \quad P(t_1) = \left(S_d \# S_1^{-1} - I_n\right) M$$

Matrix Geometric Mean

The minimal geodesic $\gamma^* : [0,1] \mapsto \mathbf{S}_n^+$ connecting $\gamma(0) = S_d$ and $\gamma(1) = S_1^{-1}$, associated with the Riemannian metric $g_A(S_d, S_1^{-1}) = \operatorname{tr}\left(A^{-1} S_d A^{-1} S_1^{-1}\right)$, is

$$\gamma^*(t) = S_d \,\#_t\, S_1^{-1} = S_d^{\frac{1}{2}} \left(S_d^{-\frac{1}{2}} S_1^{-1} S_d^{-\frac{1}{2}} \right)^t S_d^{\frac{1}{2}}$$

$$= S_1^{-1} \,\#_{1-t}\, S_d = S_1^{-\frac{1}{2}} \left(S_1^{\frac{1}{2}} S_d S_1^{\frac{1}{2}} \right)^{1-t} S_1^{-\frac{1}{2}}$$

Geometric Mean:

$$\gamma^*\left(\frac{1}{2}\right) = S_d \,\#_{\frac{1}{2}}\, S_1^{-1} = S_1^{-1} \,\#_{\frac{1}{2}}\, S_d$$

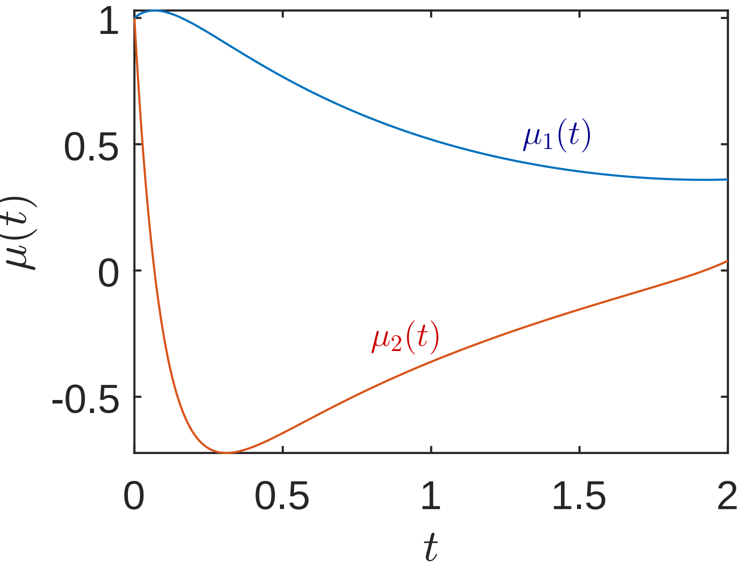
Example

$$\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \, \mathrm{d}t + \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix} \mathrm{d}w$$

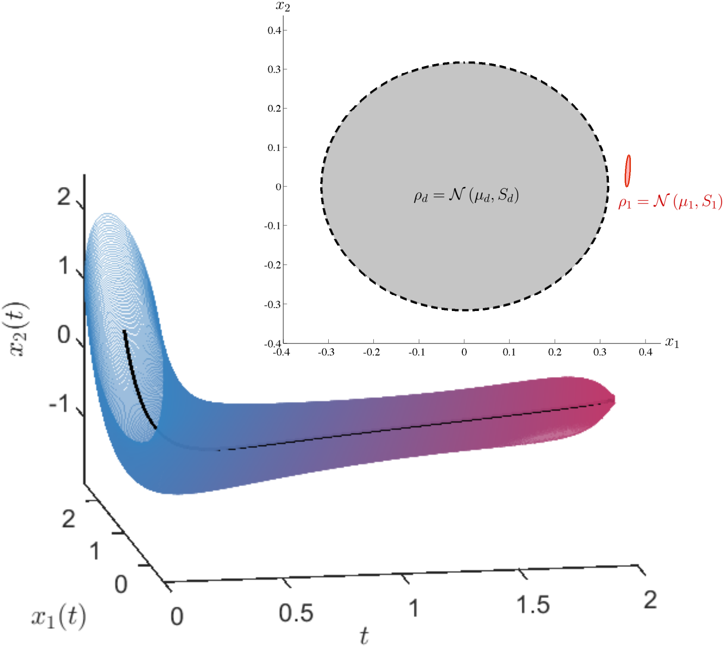
$$\rho_0 = \mathcal{N}\left((1,1)^\top, I_2\right), \quad \rho_d = \mathcal{N}\left((0,0)^\top, 0.1\, I_2\right),$$

$$Q = 100\, I_2, \quad R = 1, \quad M = I_2, \quad t_1 = 2$$

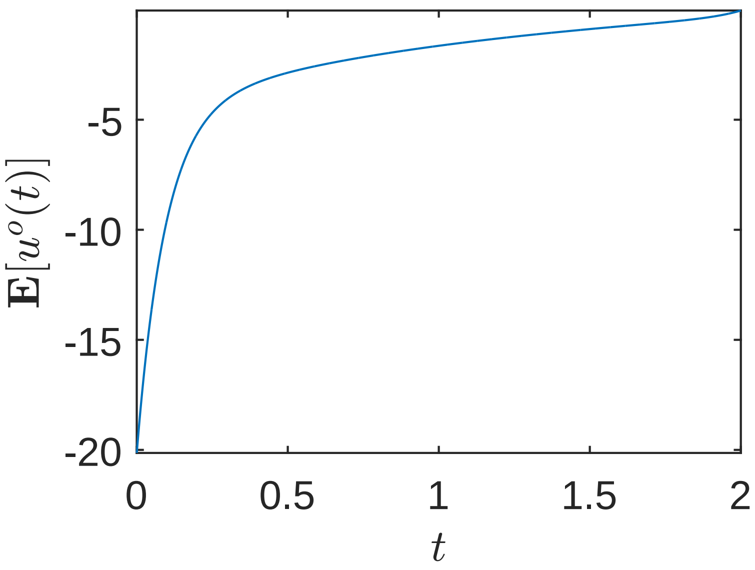
Controlled State Mean



Controlled State Covariance



Expected Optimal Control



Summary

Density regulator problem with terminal cost

LQGDR with affine state feedback

Recovers LQG/LQR as special case

Many possible extensions:

- (**conjecture**) affine state feedback is optimal
- output feedback
- geometry of different terminal costs

Thank you