Stochastic Uncertainty Propagation in Power System Dynamics using Measure-valued Proximal Recursions

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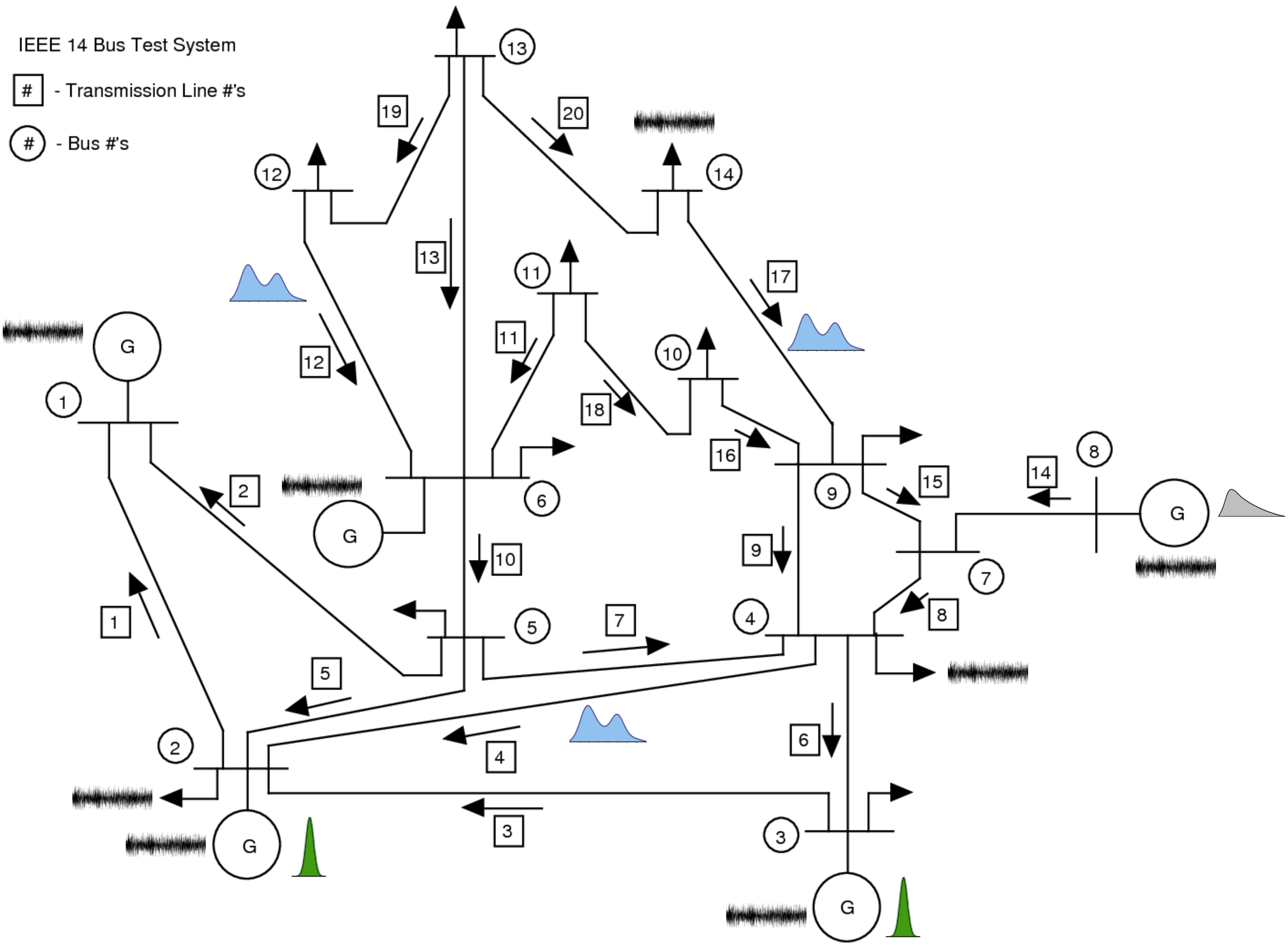
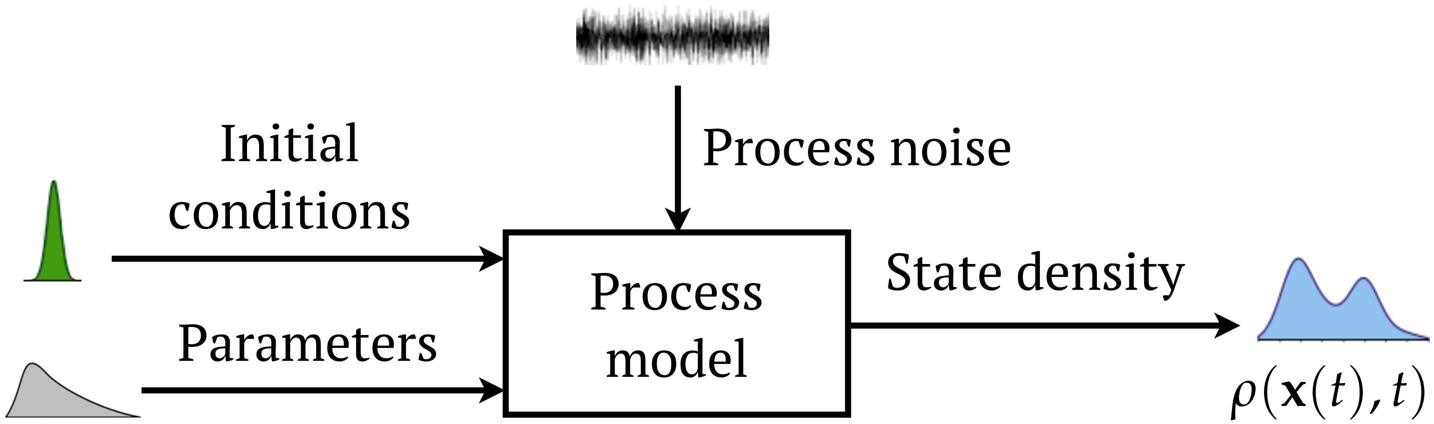
Acknowledgement: NSF Award 1923278

NSF AMPS PI Virtual Meeting, Nov. 19, 2020

Uncertainty Propagation in Power Systems



Propagating Joint Probability Density Function



Trajectory flow:

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}, t)\,\mathrm{d}t + \mathbf{g}(\mathbf{X}, t)\,\mathrm{d}\mathbf{w}(t), \quad \mathrm{d}\mathbf{w}(t) \sim \mathcal{N}(0, \mathbf{Q}\mathrm{d}t)$$

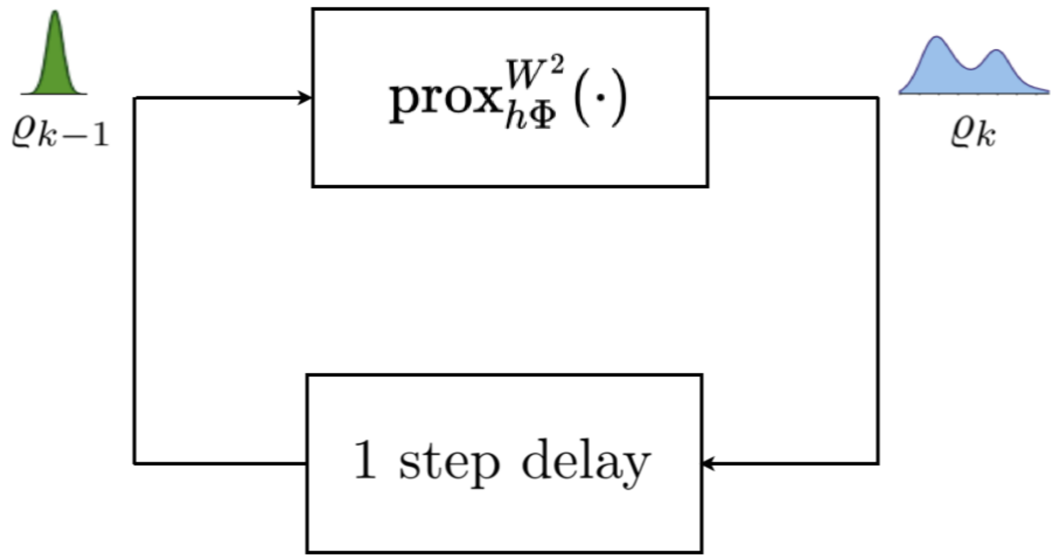
Density flow:

$$\frac{\partial \rho}{\partial t} = \mathcal{L}_{\mathrm{FP}}(\rho) := -\nabla \cdot (\rho \mathbf{f}) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\left(\mathbf{g}\mathbf{Q}\mathbf{g}^\top \right)_{ij} \rho \right)$$

What's New?

Main idea: Solve $\dfrac{\partial \rho}{\partial t} = \mathcal{L}_{\rm FP}\rho, \;\; \rho(x, t=0) = \rho_0$ as gradient flow in $\mathcal{P}_2(\mathcal{X})$

Infinite dimensional variational recursion:



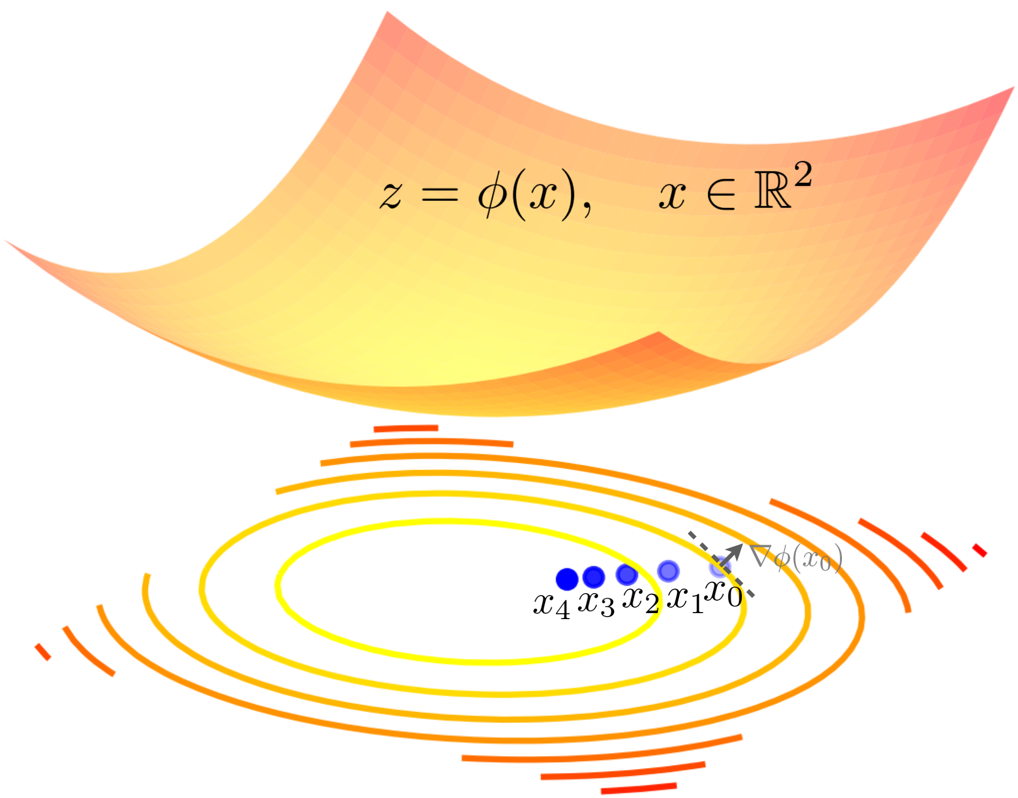
Proximal operator: $\varrho_k = \text{prox}^{W^2}_{h\Phi}(\varrho_{k-1}) := \underset{\varrho \in \mathcal{P}_2(\mathcal{X})}{\arg\inf} \left\{ \frac{1}{2} W^2(\varrho, \varrho_{k-1}) + h\Phi(\varrho) \right\}$

Optimal transport cost: $W^2(\varrho, \varrho_{k-1}) := \inf_{\pi \in \Pi(\varrho, \varrho_{k-1})} \int_{\mathcal{X} \times \mathcal{X}} c(x, y) \, \mathrm{d}\pi(x, y)$

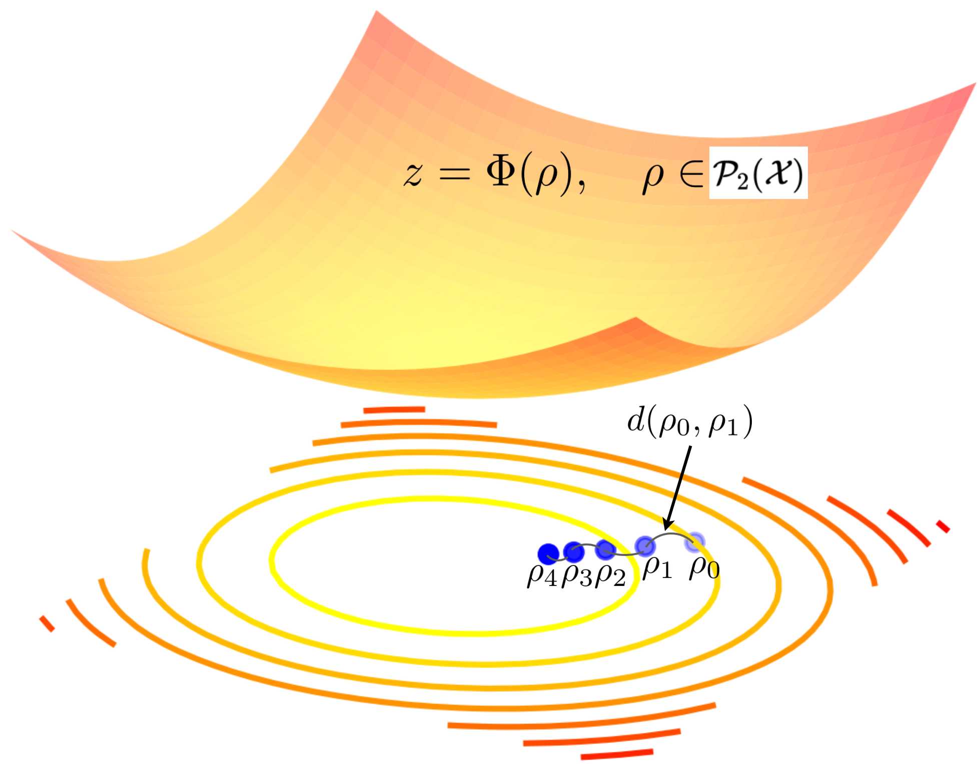
Free energy functional: $\Phi(\varrho) := \int_{\mathcal{X}} \psi \varrho \, \mathrm{d}x + \beta^{-1} \int_{\mathcal{X}} \varrho \log \varrho \, \mathrm{d}x$

Geometric Meaning of Gradient Flow

Gradient Flow in $\mathcal{X}$

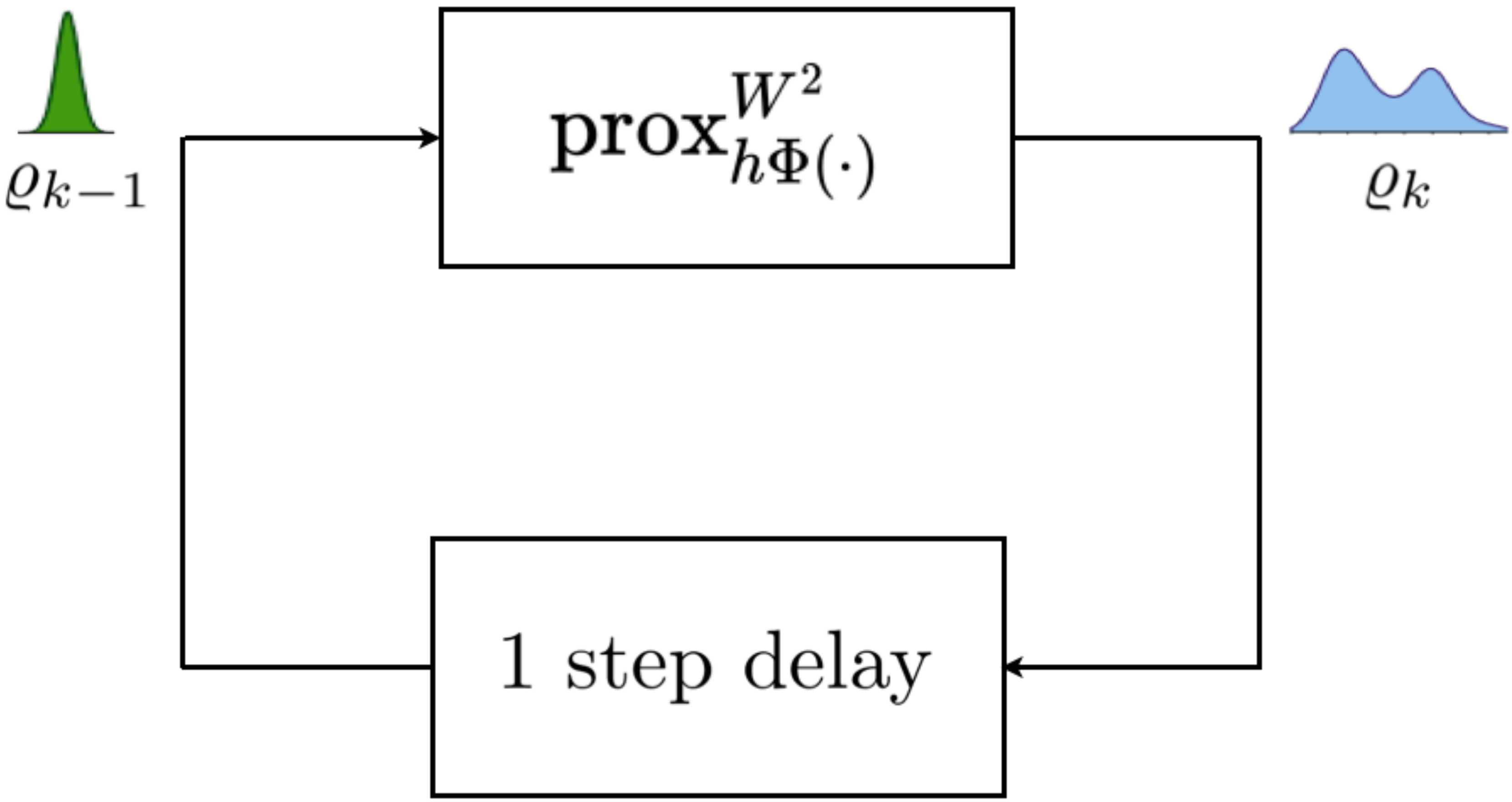


Gradient Flow in $\mathcal{P}_2(\mathcal{X})$



Algorithm: Gradient Ascent on the Dual Space

Uncertainty propagation via point clouds



No spatial discretization or function approximation

Algorithm: Gradient Ascent on the Dual Space

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \psi \rho) + \beta^{-1} \Delta \rho$$

$\Updownarrow$ **Proximal Recursion**

$$\rho_k = \rho(\boldsymbol{x}, t = kh) = \underset{\rho \in \mathcal{P}_2(\mathbb{R}^n)}{\arg\inf} \left\{ \frac{1}{2} W^2(\rho, \rho_{k-1}) + h\, \Phi(\rho) \right\}$$

$\Downarrow$ **Discrete Primal Formulation**

$$\boldsymbol{\varrho}_k = \arg\min_{\boldsymbol{\varrho}} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + h \left\langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \right\rangle \right\}$$

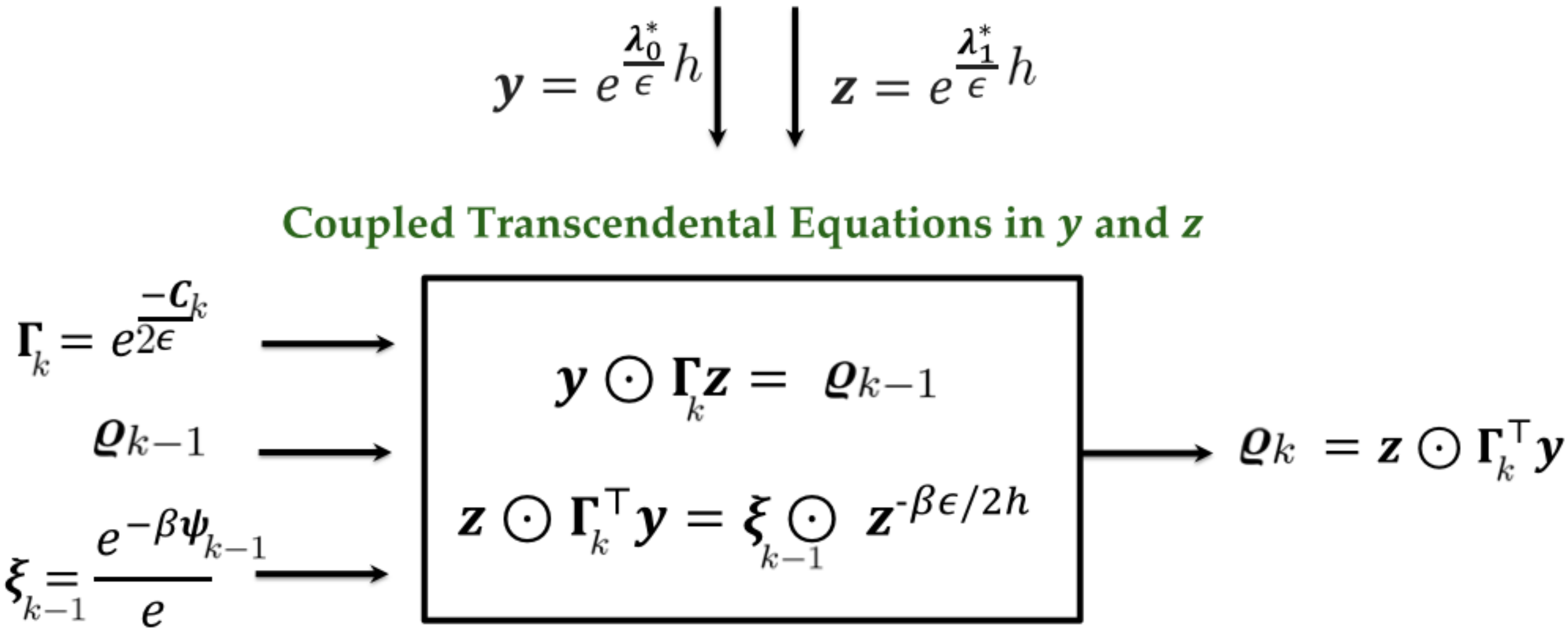
$\Downarrow$ **Entropic Regularization**

$$\boldsymbol{\varrho}_k = \arg\min_{\boldsymbol{\varrho}} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + \epsilon H(\boldsymbol{M}) + h \left\langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \right\rangle \right\}$$

$\Updownarrow$ **Dualization**

$$\boldsymbol{\lambda}_0^{\text{opt}}, \boldsymbol{\lambda}_1^{\text{opt}} = \underset{\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1 \geq 0}{\arg\max} \Bigg\{ \langle \boldsymbol{\lambda}_0, \boldsymbol{\varrho}_{k-1} \rangle - F^{\star}(-\boldsymbol{\lambda}_1) - \frac{\epsilon}{h} \left(\exp(\boldsymbol{\lambda}_0^{\top} h/\epsilon) \exp(-\boldsymbol{C}_k/2\epsilon) \exp(\boldsymbol{\lambda}_1 h/\epsilon) \right) \Bigg\}$$

Recursion on the Cone

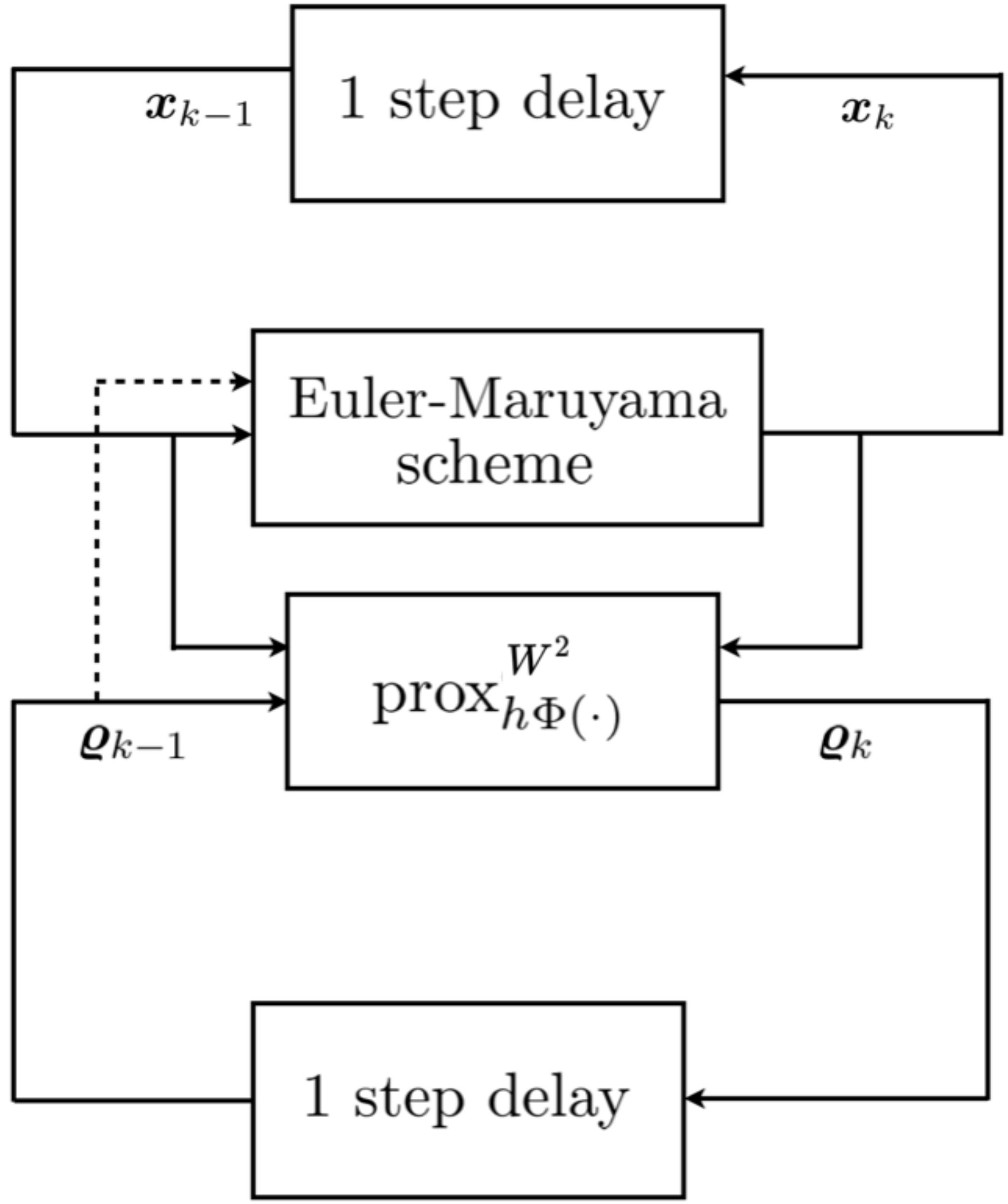


Theorem: Consider the recursion on the cone $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0}$

$$\boldsymbol{y} \odot (\boldsymbol{\Gamma}_k \boldsymbol{z}) = \varrho_{k-1}, \quad \boldsymbol{z} \odot \left(\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}\right) = \boldsymbol{\xi}_{k-1} \odot \boldsymbol{z}^{-\frac{\beta\epsilon}{h}},$$

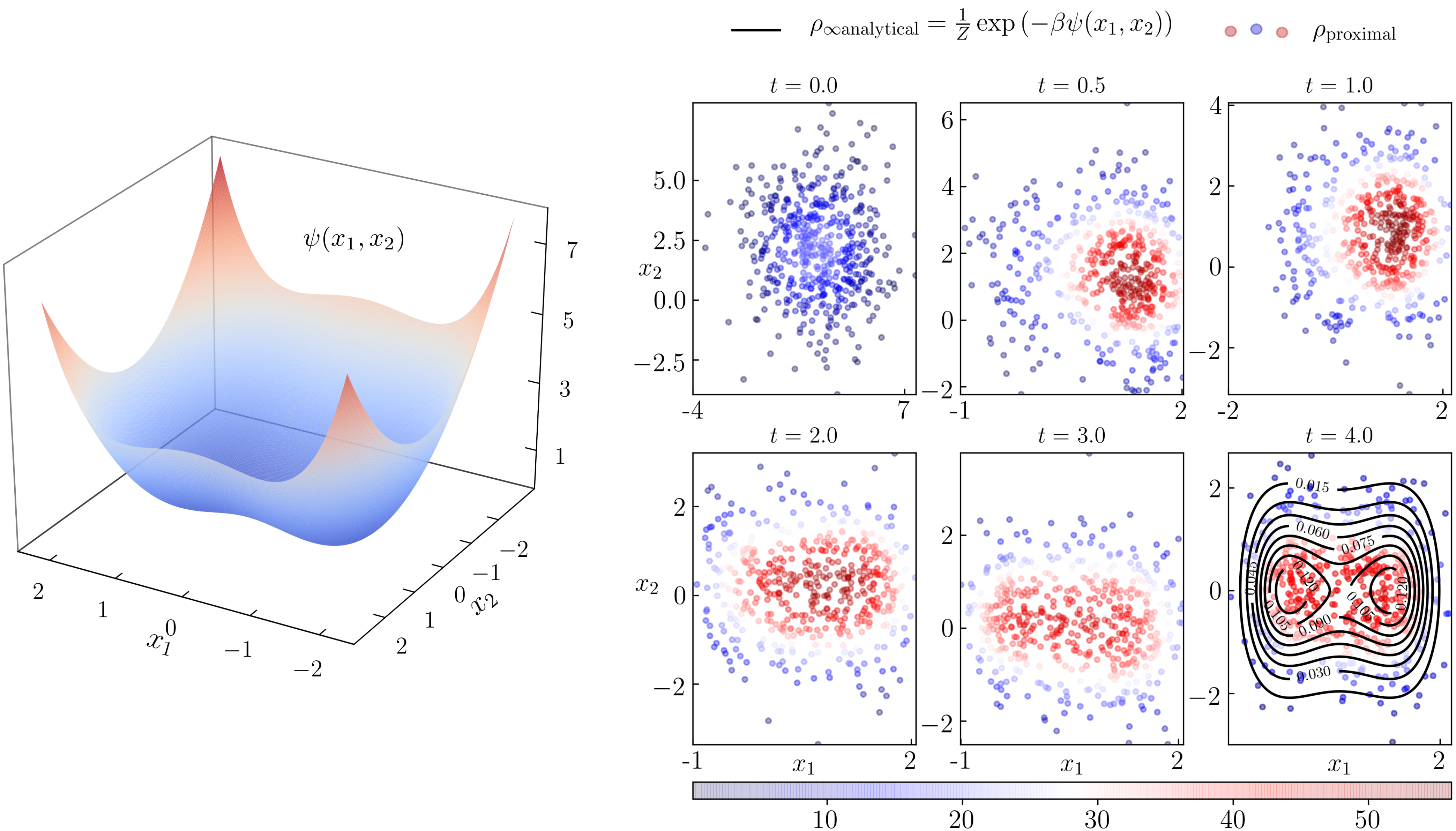
Then the solution $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ gives the proximal update $\varrho_k = \boldsymbol{z}^* \odot (\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}^*)$

Algorithmic Setup

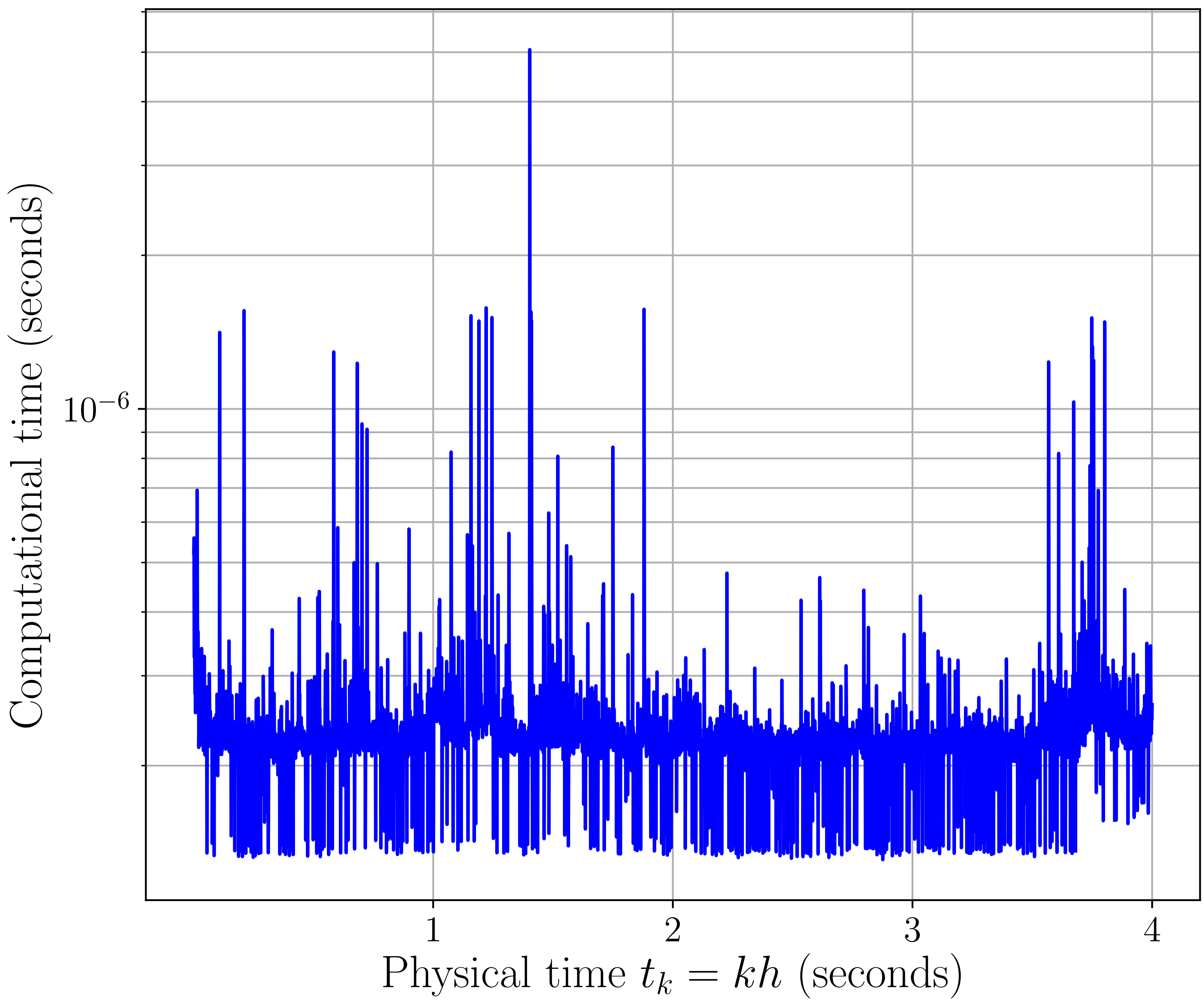


Theorem: Block co-ordinate iteration of $(\mathbf{y}, \mathbf{z})$ recursion is contractive on $\mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0}$.

Proximal Prediction: Nonlinear Non-Gaussian



Computational Time: Nonlinear Non-Gaussian



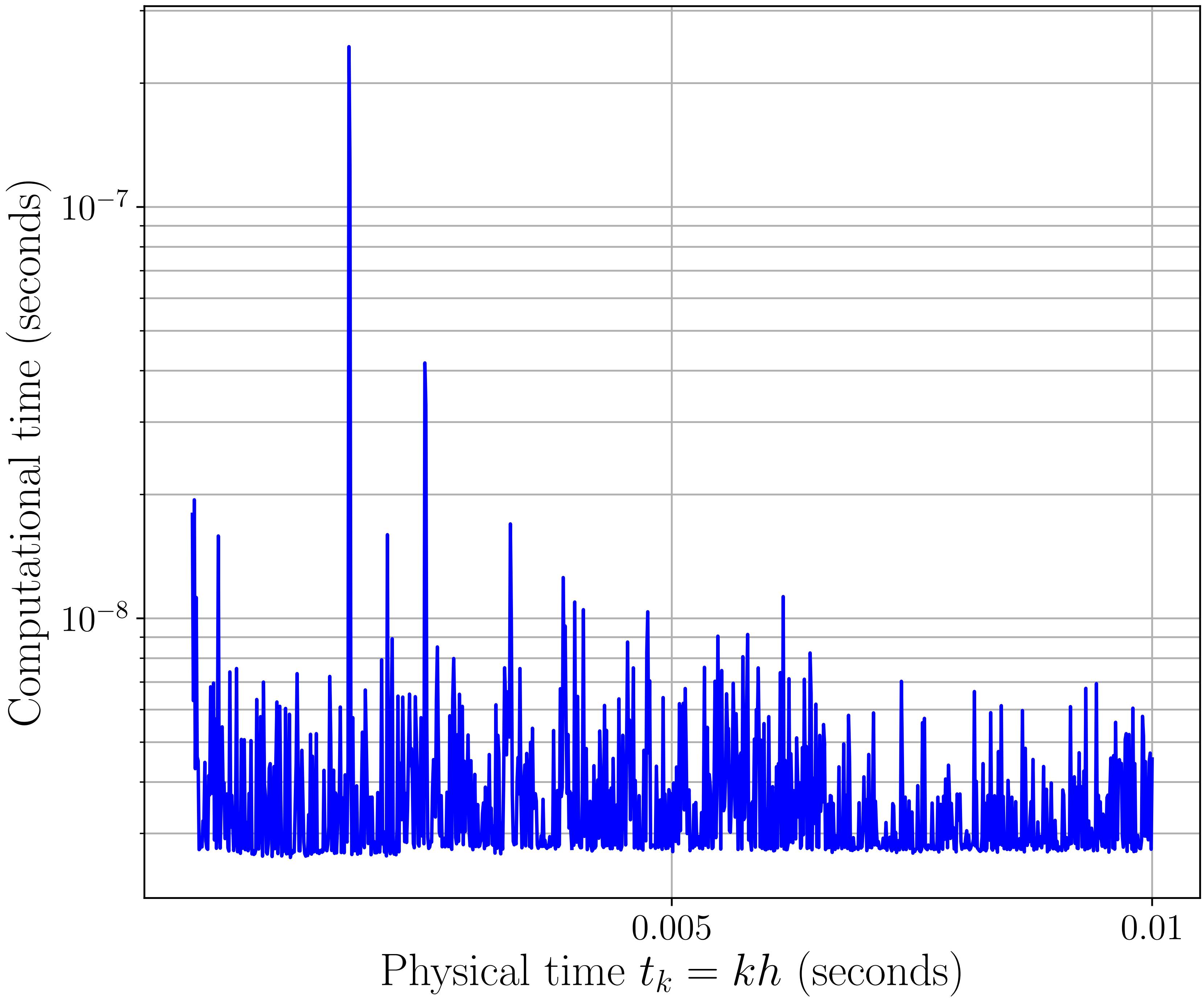
Proximal Prediction: Satellite in Geocentric Orbit

Here, $\mathcal{X} \equiv \mathbb{R}^6$

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \\ \mathrm{d}v_x \\ \mathrm{d}v_y \\ \mathrm{d}v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ -\dfrac{\mu x}{r^3} + (f_x)_{\text{pert}} - \gamma v_x \\ -\dfrac{\mu y}{r^3} + (f_y)_{\text{pert}} - \gamma v_y \\ -\dfrac{\mu z}{r^3} + (f_z)_{\text{pert}} - \gamma v_z \end{pmatrix} \mathrm{d}t + \sqrt{2\beta^{-1}\gamma} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{d}w_1 \\ \mathrm{d}w_2 \\ \mathrm{d}w_3 \end{pmatrix},$$

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}_{\text{pert}} = \begin{pmatrix} s\theta\ c\phi & c\theta\ c\phi & -s\phi \\ s\theta\ s\phi & c\theta\ s\phi & c\phi \\ c\theta & -s\theta & 0 \end{pmatrix} \begin{pmatrix} \dfrac{k}{2r^4}\left(3(s\theta)^2 - 1\right) \\ -\dfrac{k}{r^5} s\theta\ c\theta \\ 0 \end{pmatrix}, k := 3J_2R_{\mathrm{E}}^2, \mu = \text{constant}$$

Computational Time: Satellite in Geocentric Orbit



Network Reduced Power System Model

Structure preserving model:

$$m_i\ddot{\theta}_i + \gamma_i\dot{\theta}_i = P_i^{\text{mech}} - \sum_{j=1}^{n} k_{ij}\sin(\theta_i - \theta_j + \varphi_{ij}) + \sigma_i \times \text{stochastic forcing}, \; i = 1, \ldots, n$$

Define positive diagonal matrices:

$$\boldsymbol{M} := \text{diag}(m_1, \ldots, m_n), \quad \boldsymbol{\Gamma} := \text{diag}(\gamma_1, \ldots, \gamma_n), \quad \boldsymbol{\Sigma} := \text{diag}(\sigma_1, \ldots, \sigma_n)$$

Mixed conservative-dissipative SDE in state $(\boldsymbol{\theta}, \boldsymbol{\omega}) \in \mathbb{T}^n \times \mathbb{R}^n$:

$$\begin{pmatrix} \mathrm{d}\boldsymbol{\theta} \\ \mathrm{d}\boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega} \\ -\boldsymbol{M}^{-1}\nabla_{\boldsymbol{\theta}}V(\boldsymbol{\theta}) - \boldsymbol{M}^{-1}\boldsymbol{\Gamma}\boldsymbol{\omega} \end{pmatrix}\mathrm{d}t + \underbrace{\begin{pmatrix} \boldsymbol{0}_{n\times n} \\ \boldsymbol{M}^{-1}\boldsymbol{\Sigma} \end{pmatrix}\mathrm{d}\boldsymbol{w}}_{\text{anisotropic degenerate diffusion}} \quad , \; \boldsymbol{w} \in \mathbb{R}^n$$

Potential function $V : \mathbb{T}^n \mapsto \mathbb{R}_{\geq 0}$

$$V(\boldsymbol{\theta}) := \sum_{i=1}^{n} P_i^{\text{mech}}\theta_i + \sum_{(i,j)\in\mathcal{E}} k_{ij}(1 - \cos(\theta_i - \theta_j + \varphi_{ij}))$$

Transform to isotropic degenerate diffusion

Pushforward joint PDF via invertible linear map $\boldsymbol{\Psi}$:

$$\rho(\boldsymbol{\theta}, \boldsymbol{\omega}) \mapsto \tilde{\rho}(\boldsymbol{\xi}, \boldsymbol{\eta}), \quad \tilde{\rho} = \boldsymbol{\Psi}_{\sharp}\, \rho$$

where

$$\begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} = \underbrace{\left(\boldsymbol{I}_2 \otimes \boldsymbol{M}\boldsymbol{\Sigma}^{-1}\right)}_{\boldsymbol{\Psi}} \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix}$$

By Ito's lemma:

$$\begin{pmatrix} \mathrm{d}\boldsymbol{\xi} \\ \mathrm{d}\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\eta} \\ -\nabla_{\boldsymbol{\xi}} U(\boldsymbol{\xi}) - \nabla_{\boldsymbol{\eta}} F(\boldsymbol{\eta}) \end{pmatrix} \mathrm{d}t \; + \; \begin{pmatrix} \boldsymbol{0}_{n\times n} \\ \boldsymbol{I}_n \end{pmatrix} \mathrm{d}\boldsymbol{w}$$

where the new potentials:

$$U(\boldsymbol{\xi}) := \sum_{i=1}^{n} \frac{1}{\sigma_i} P_i^{\text{mech}} \xi_i + \sum_{(i,j)\in\mathcal{E}} \frac{m_i}{\sigma_i^2} k_{ij} \left(1 - \cos\left(\frac{\sigma_i}{m_i}\xi_i - \frac{\sigma_j}{m_j}\xi_j + \varphi_{ij}\right)\right)$$

$$F(\boldsymbol{\eta}) := \frac{1}{2}\langle \boldsymbol{\eta}, \boldsymbol{M}^{-1}\boldsymbol{\Gamma}\boldsymbol{\eta}\rangle$$

Propagate the Pushforward

Kinetic Fokker-Planck PDE for the pushforward:

$$\frac{\partial \tilde{\rho}}{\partial t} = -\langle \boldsymbol{\eta}, \nabla_{\boldsymbol{\xi}} \tilde{\rho} \rangle + \nabla_{\boldsymbol{\eta}} \cdot ((\nabla_{\boldsymbol{\xi}} U(\boldsymbol{\xi}) + \nabla_{\boldsymbol{\eta}} F(\boldsymbol{\eta}))\tilde{\rho}) + \frac{1}{2}\Delta_{\boldsymbol{\eta}} \tilde{\rho}$$

The following is a Lyapunov functional

$$\Phi(\tilde{\rho}) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \left(U(\boldsymbol{\xi}) + F(\boldsymbol{\eta}) + \frac{1}{2} \log \tilde{\rho} \right) \tilde{\rho} \, \mathrm{d}\boldsymbol{\xi} \mathrm{d}\boldsymbol{\eta}$$

but the PDE cannot be written as gradient flow of Φ w.r.t. W metric!!!

Proximal Update

Instead, do the proximal recursion:

$$\tilde{\varrho}_k = \mathrm{prox}_{h\widetilde{\Phi}}^{\widetilde{W}}(\tilde{\varrho}_{k-1}), \quad k \in \mathbb{N}$$

with pair $\left(\widetilde{W}, \widetilde{\Phi}\right)$ given by

$$\widetilde{W}^2(\tilde{\varrho}, \tilde{\varrho}_{k-1}) = \inf_{\pi \in \Pi(\tilde{\varrho}, \tilde{\varrho}_{k-1})} \int_{\mathbb{T}^{2n} \times \mathbb{R}^{2n}} \left\{ \left\| \bar{\boldsymbol{\eta}} - \boldsymbol{\eta} + h \nabla_{\boldsymbol{\xi}} U(\boldsymbol{\xi}) \right\|_2^2 + 12 \left\| \frac{\bar{\boldsymbol{\xi}} - \boldsymbol{\xi}}{h} - \frac{\bar{\boldsymbol{\eta}} + \boldsymbol{\eta}}{2} \right\|_2^2 \right\} \mathrm{d}\pi(\boldsymbol{\xi}, \boldsymbol{\eta}, \bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}})$$

$$\widetilde{\Phi}(\tilde{\rho}) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \left(F(\boldsymbol{\eta}) + \frac{1}{2} \log \tilde{\rho} \right) \tilde{\rho} \, \mathrm{d}\boldsymbol{\xi} \mathrm{d}\boldsymbol{\eta}$$

Guarantee: $\tilde{\rho}_k \rightarrow \tilde{\rho} \quad \text{as } h \downarrow 0$

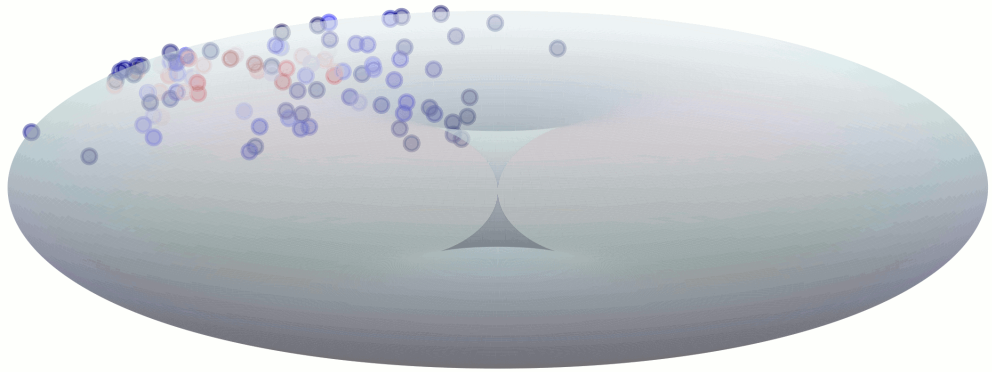
Finally, come back to original coordinates:

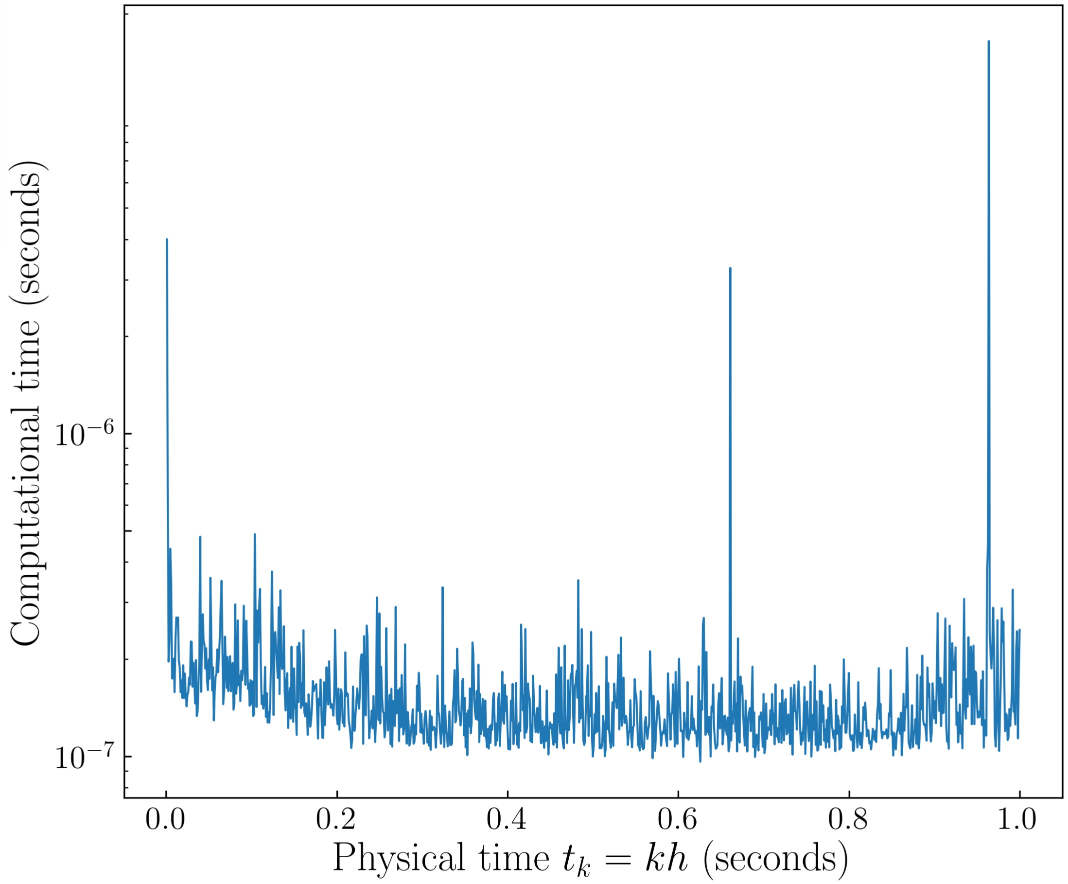
$$\rho(\boldsymbol{\theta}, \boldsymbol{\omega}, t) = \underbrace{\det(\boldsymbol{\Psi})}_{\left(\prod_{i=1}^{n} m_i/\sigma_i\right)^2} \tilde{\rho}(\boldsymbol{M}\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}, \boldsymbol{M}\boldsymbol{\Sigma}^{-1}\boldsymbol{\omega}, t)$$

Proximal Prediction: Power System with $n = 2$

Projection of the joint PDF on $\mathbb{T}^2$

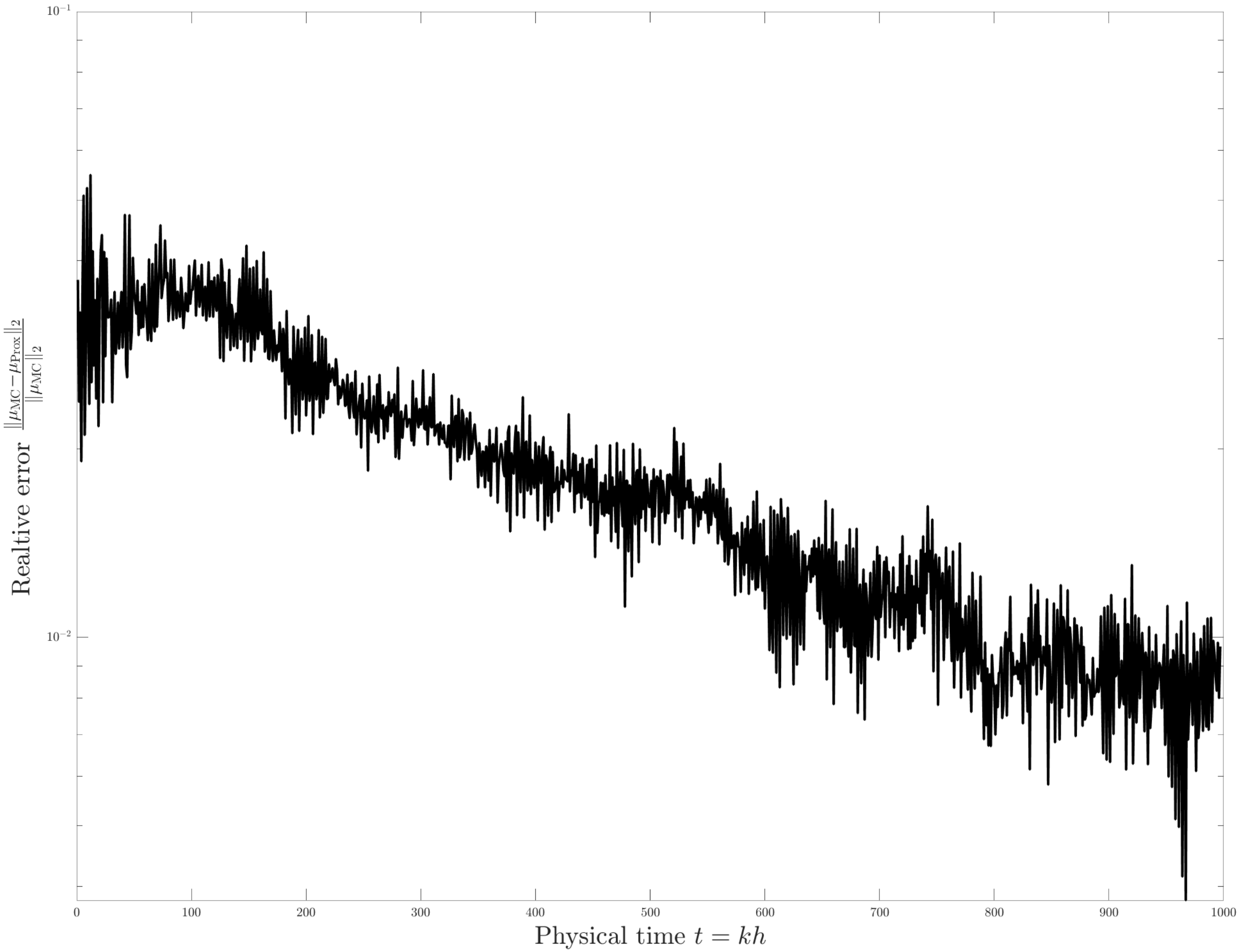
$\mathrm{t} = 0.0000\ \mathrm{s}$





Proximal Prediction: Power System with $n = 20$

Randomly generated parameters using interval data from:
Dorfler, F., and Bullo, F., Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators, *SIAM J. Control and Optimization*, Vol. 50, No. 3, pp. 1616–1642, 2012.



Proximal Prediction: Power System with $n = 20$

Randomly generated parameters using interval data from:
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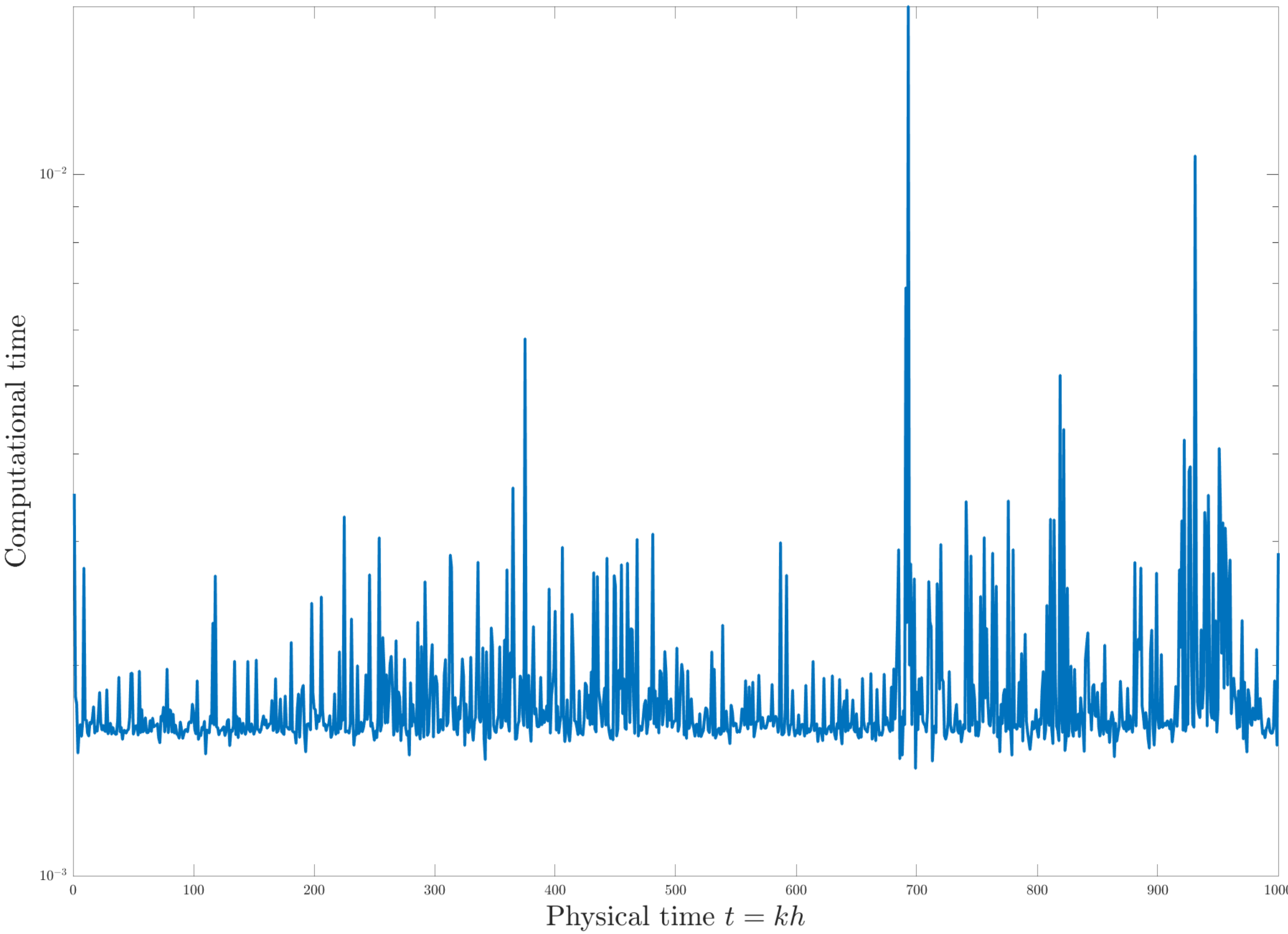

Summary

Fast proximal recursions for PDF propagation in power systems

Ongoing

Large scale implementation: ~1000 generators in ~seconds

Control of joint PDFs via state feedback

Thank You

Projection of the joint PDF on $\mathbb{R}^2$