Gradient Flows for Prediction and Control of Densities

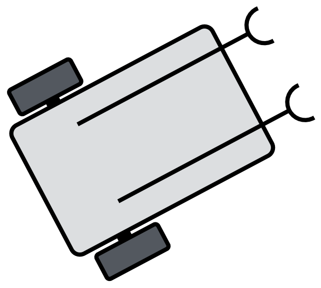
Abhishek Halder

Department of Applied Mathematics
University of California, Santa Cruz
Santa Cruz, CA 95064

Joint work with Kenneth F. Caluya (UC Santa Cruz)
and Tryphon T. Georgiou (UC Irvine)

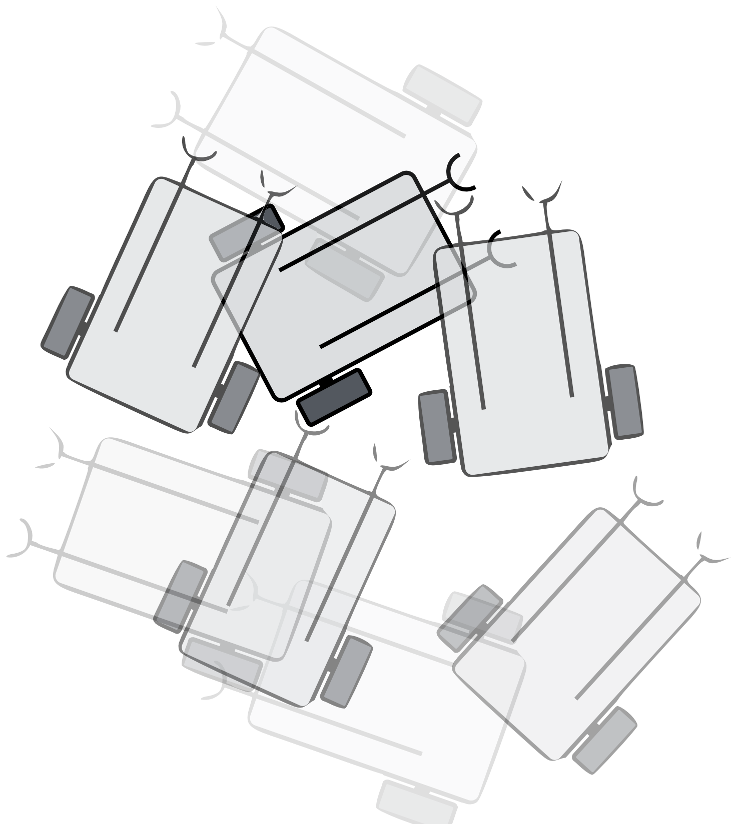
What is density?

Probability Density Fn.



$$x(t) \in \mathcal{X} \equiv \mathbb{R}^2 \times \mathbb{S}^1$$

Probability Density Fn.

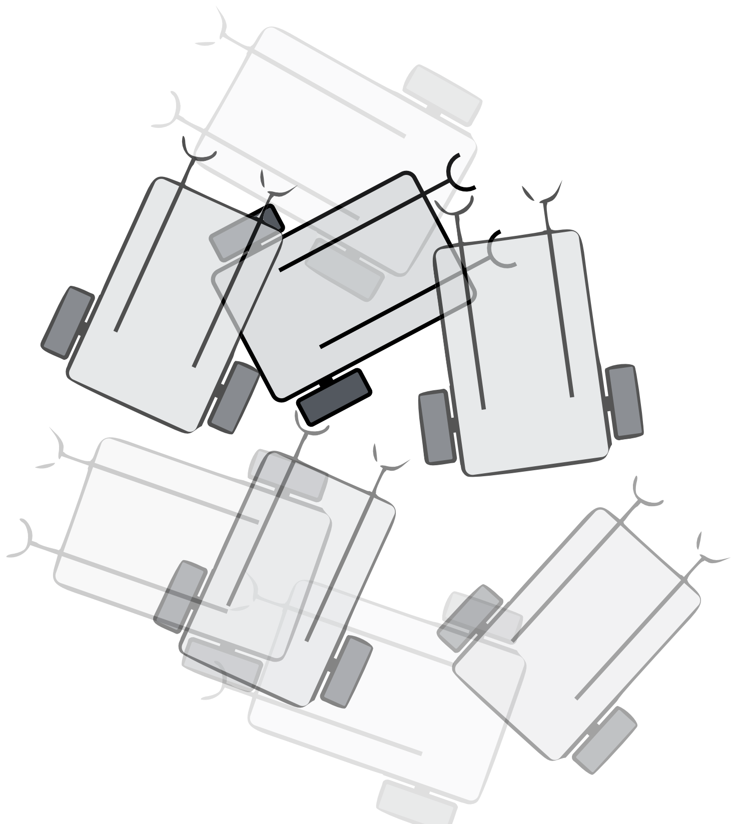


$$x(t) \in \boldsymbol{\mathcal{X}} \equiv \mathbb{R}^2 \times \mathbb{S}^1$$

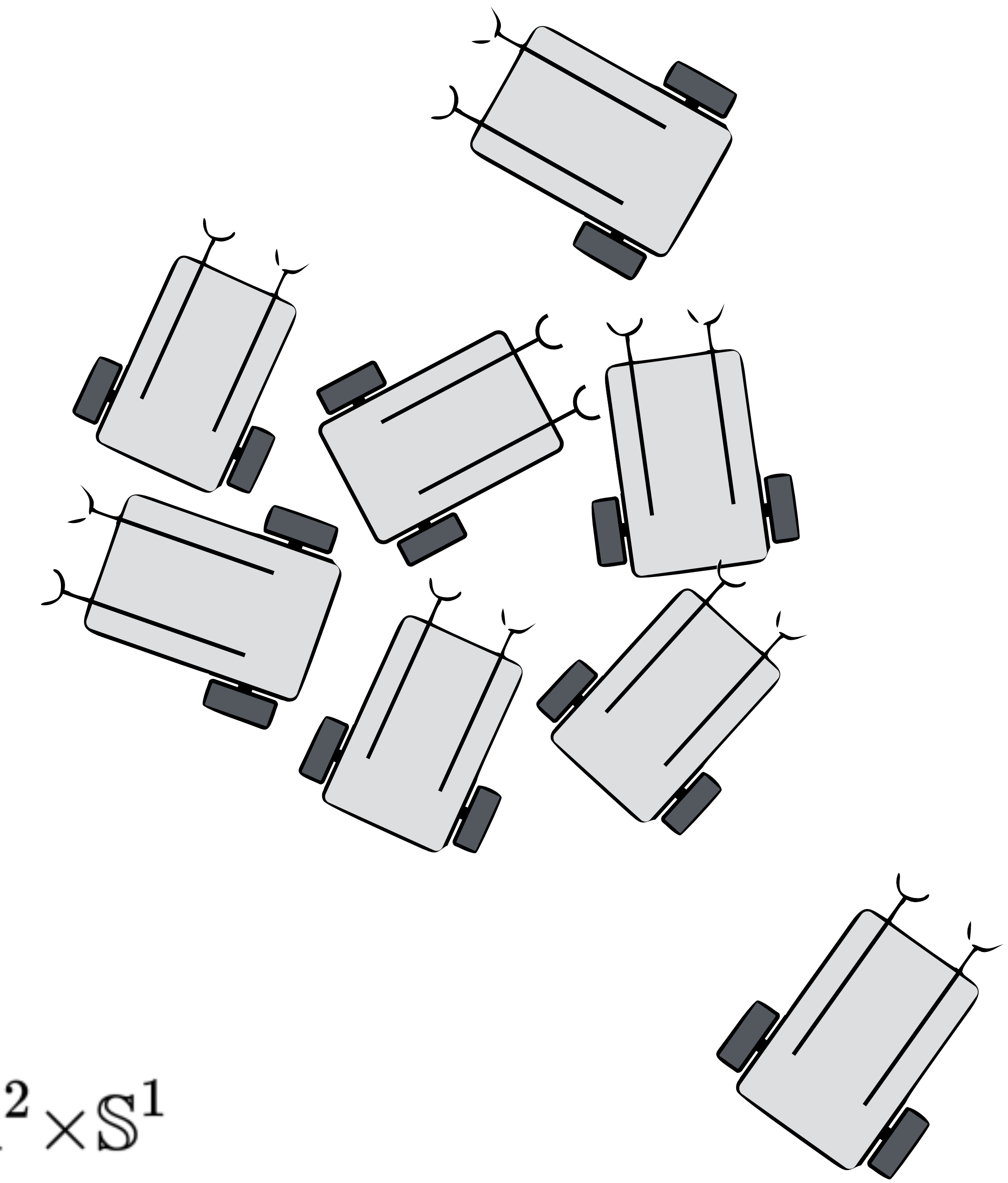
$$\rho(x,t) : \boldsymbol{\mathcal{X}} \times [0,\infty) \mapsto \mathbb{R}_{\geq 0}$$

$$\int_{\mathcal{X}} \rho \,\mathrm{d}x = 1 \quad \text{for all} \quad t \in [0,\infty)$$

Probability Density Fn.



Population Density Fn.



$$x(t) \in \boldsymbol{\mathcal{X}} \equiv \mathbb{R}^2 \times \mathbb{S}^1$$

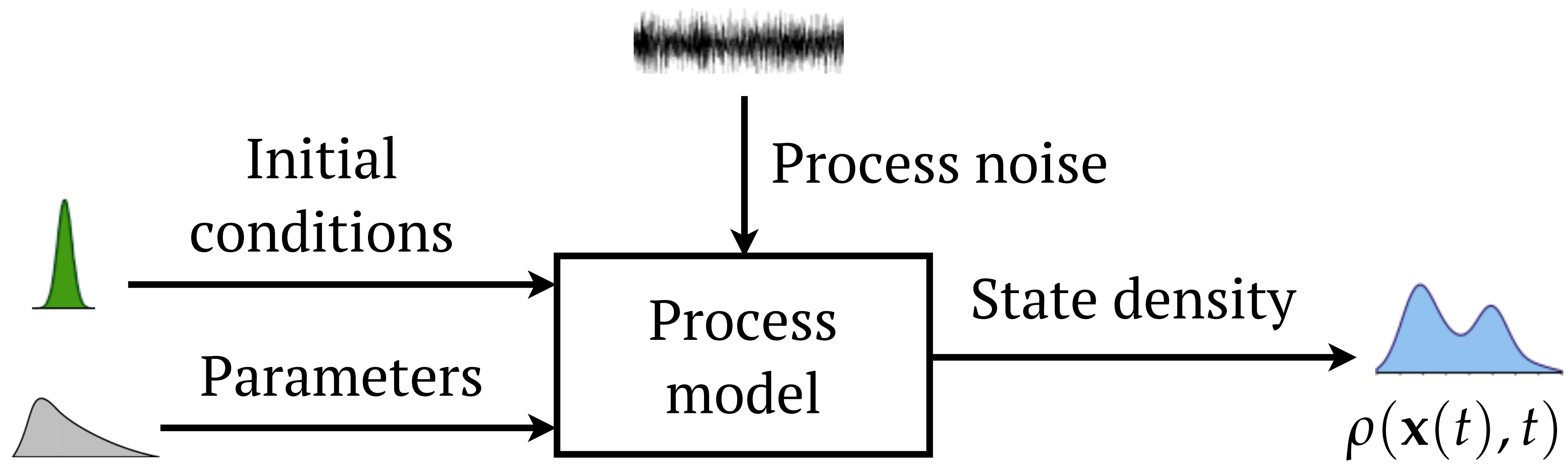
$$\rho(x,t) : \boldsymbol{\mathcal{X}} \times [0,\infty) \mapsto \mathbb{R}_{\geq 0}$$

$$\int_{\mathcal{X}} \rho \, \mathrm{d}x = 1 \quad \text{for all} \quad t \in [0,\infty)$$

Why bother about densities?

Prediction Problem

Compute
joint state PDF
$\rho(x,t)$



Trajectory flow:

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X},t)\,\mathrm{d}t + \mathbf{g}(\mathbf{X},t)\,\mathrm{d}\mathbf{w}(t), \quad \mathrm{d}\mathbf{w}(t) \sim \mathcal{N}(0,\mathbf{Q}\mathrm{d}t)$$

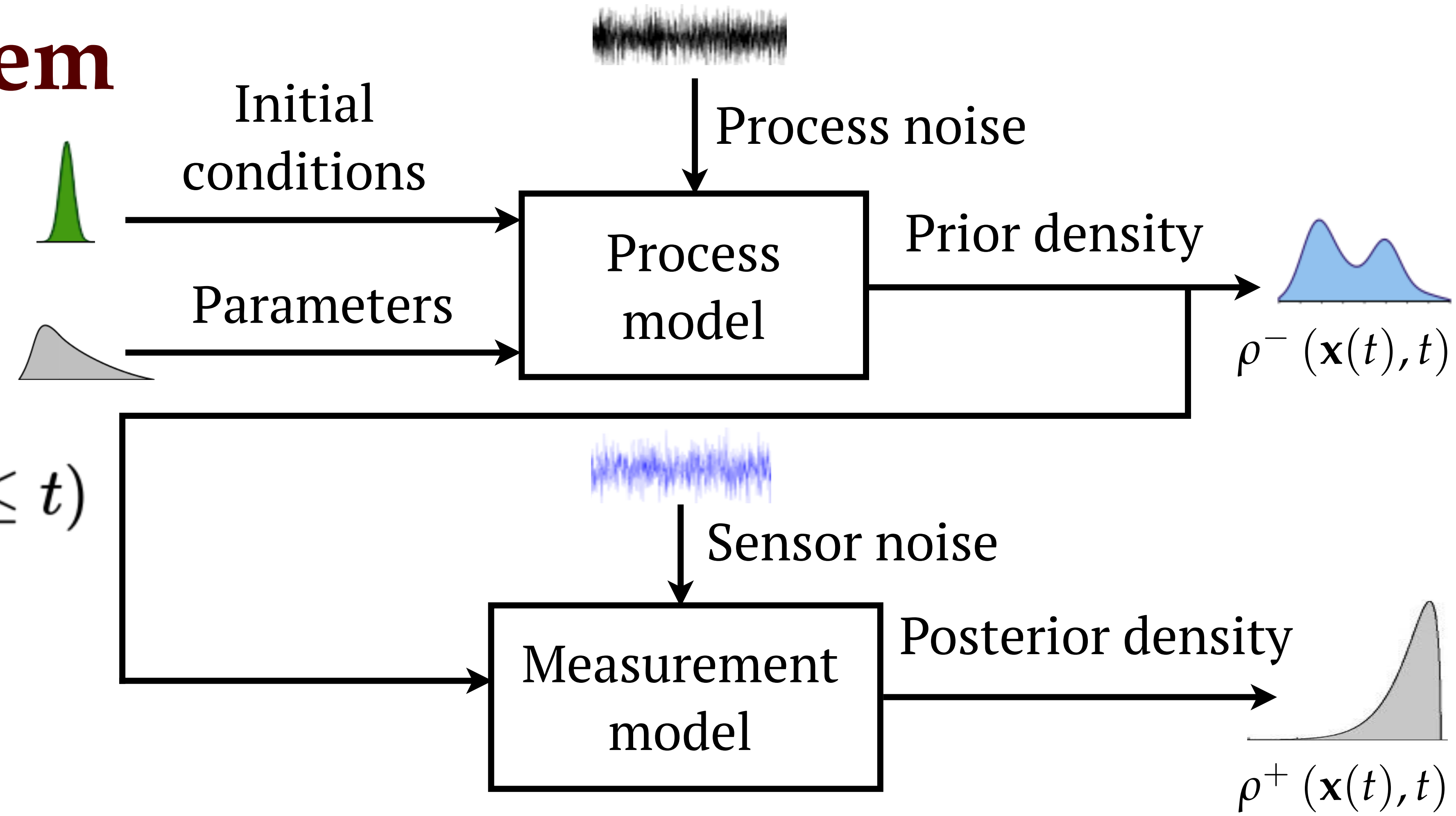
Density flow:

$$\frac{\partial\rho}{\partial t} = \mathcal{L}_{\mathrm{FP}}(\rho) := -\nabla\cdot(\rho\mathbf{f}) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\left(\left(\mathbf{g}\mathbf{Q}\mathbf{g}^{\top}\right)_{ij}\rho\right)$$

Filtering Problem

Compute conditional joint state PDF

$$\rho^{+} \equiv \rho(x, t | z(s), 0 \leq s \leq t)$$



Trajectory flow:

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}, t)\,\mathrm{d}t + \mathbf{g}(\mathbf{X}, t)\,\mathrm{d}\mathbf{w}(t), \qquad \mathrm{d}\mathbf{w}(t) \sim \mathcal{N}(0, \mathbf{Q}\mathrm{d}t)$$

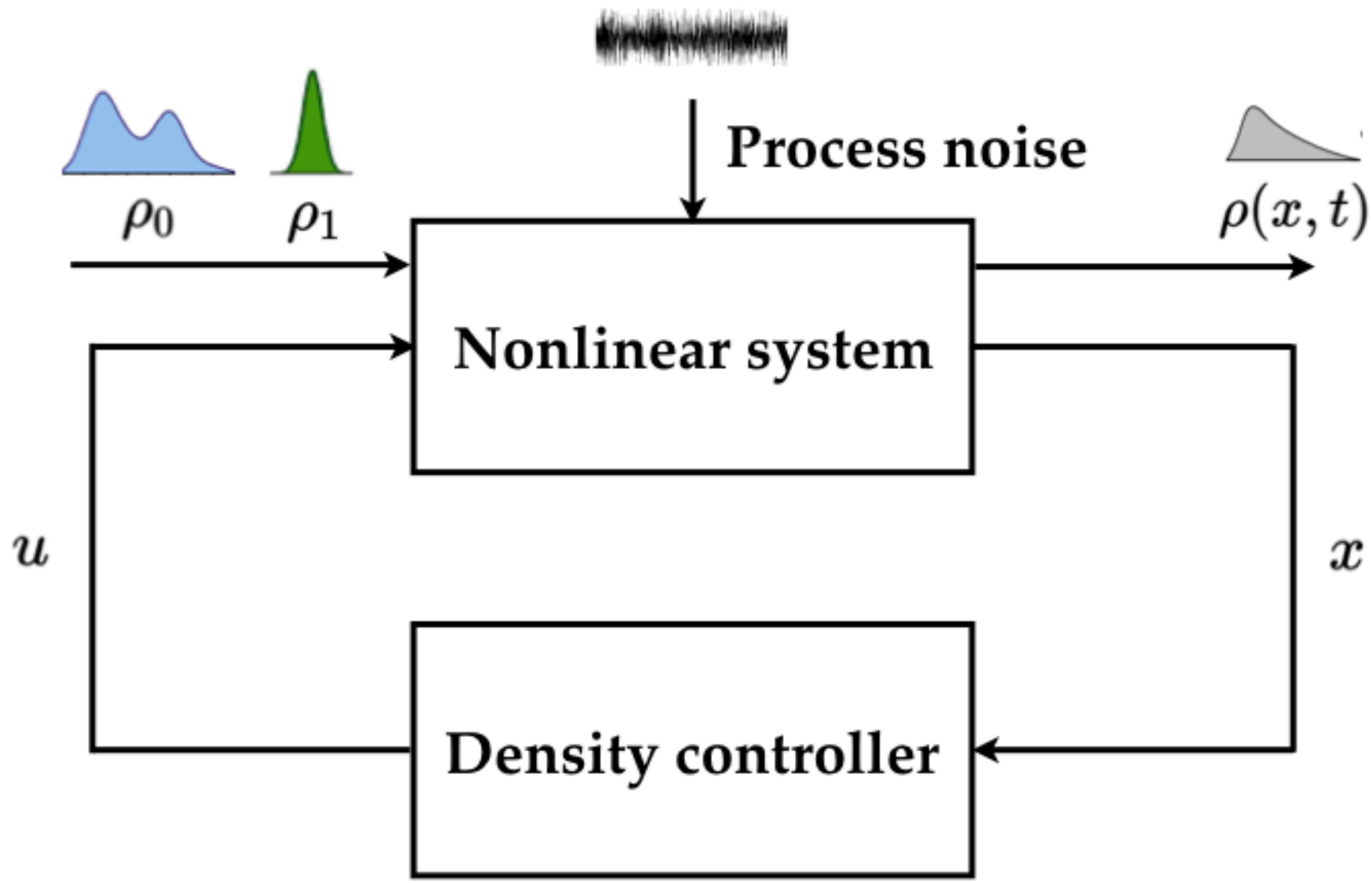
$$\mathrm{d}\mathbf{Z}(t) = \mathbf{h}(\mathbf{X}, t)\,\mathrm{d}t + \mathrm{d}\mathbf{v}(t), \qquad \mathrm{d}\mathbf{v}(t) \sim \mathcal{N}(0, \mathbf{R}\mathrm{d}t)$$

Density flow:

$$\mathrm{d}\rho^{+} = \left[\mathcal{L}_{\mathrm{FP}}\mathrm{d}t + \left(\mathbf{h}(\mathbf{x}, t) - \mathbb{E}_{\rho^{+}}\{\mathbf{h}(\mathbf{x}, t)\}\right)^{\top}\mathbf{R}^{-1}\left(\mathrm{d}\mathbf{z}(t) - \mathbb{E}_{\rho^{+}}\{\mathbf{h}(\mathbf{x}, t)\}\mathrm{d}t\right)\right]\rho^{+}$$

Control Problem

Steer joint state PDF
via feedback control



$$\underset{u\in\mathcal{U}}{\text{minimize}}\ \mathbb{E}\left[\int_0^1 \|u\|_2^2 \,\mathrm{d}t\right]$$

subject to

$$\mathrm{d}x = f(x,u,t)\,\mathrm{d}t \ + \ g(x,t)\,\mathrm{d}w,$$

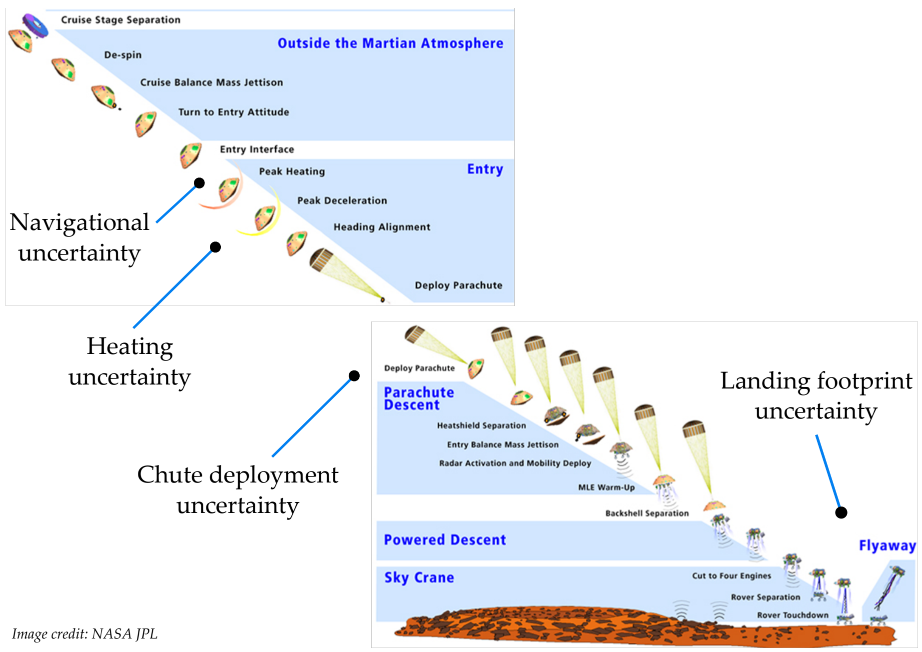
$$x(t=0)\sim\rho_0, \quad x(t=1)\sim\rho_1$$

PDFs in Mars Entry-Descent-Landing

Prediction Problem

Filtering Problem

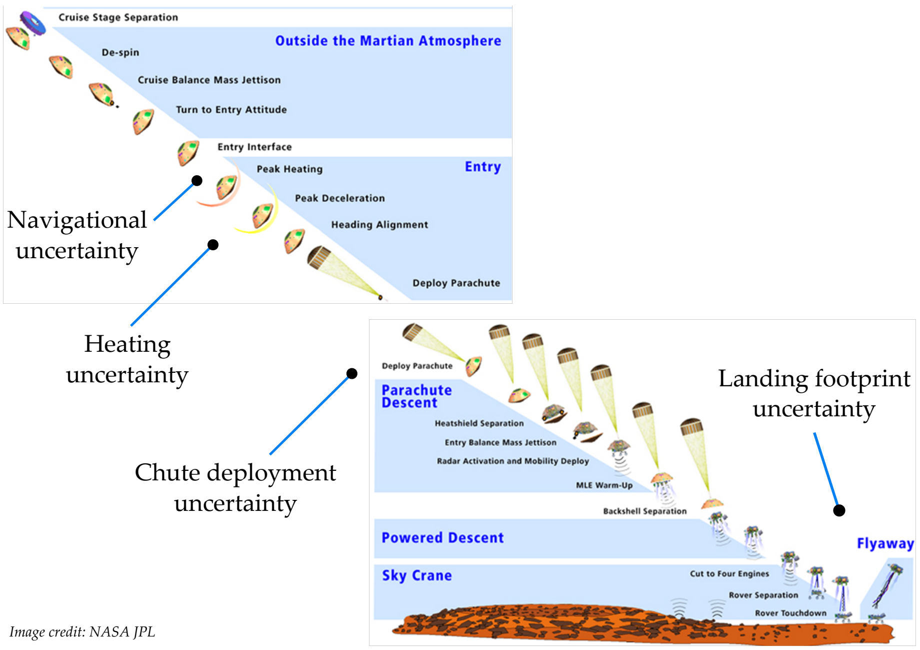
Control Problem



Predict heating rate uncertainty

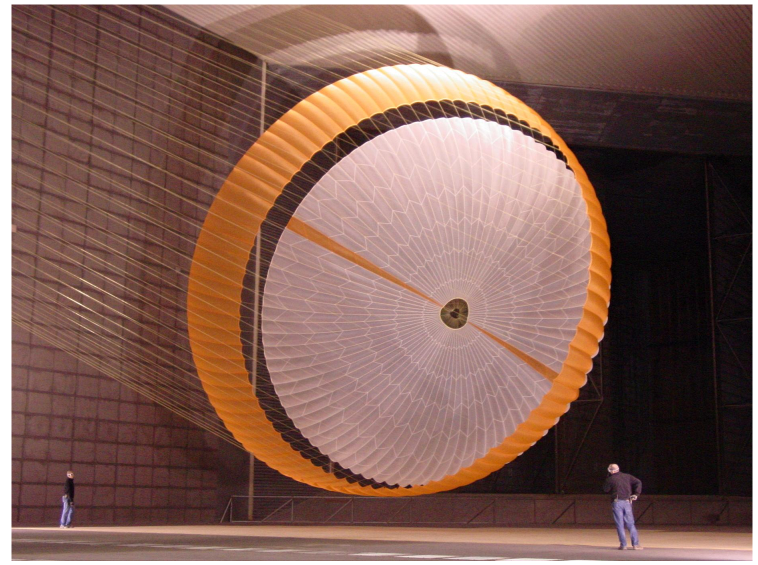
PDFs in Mars Entry-Descent-Landing

Prediction Problem

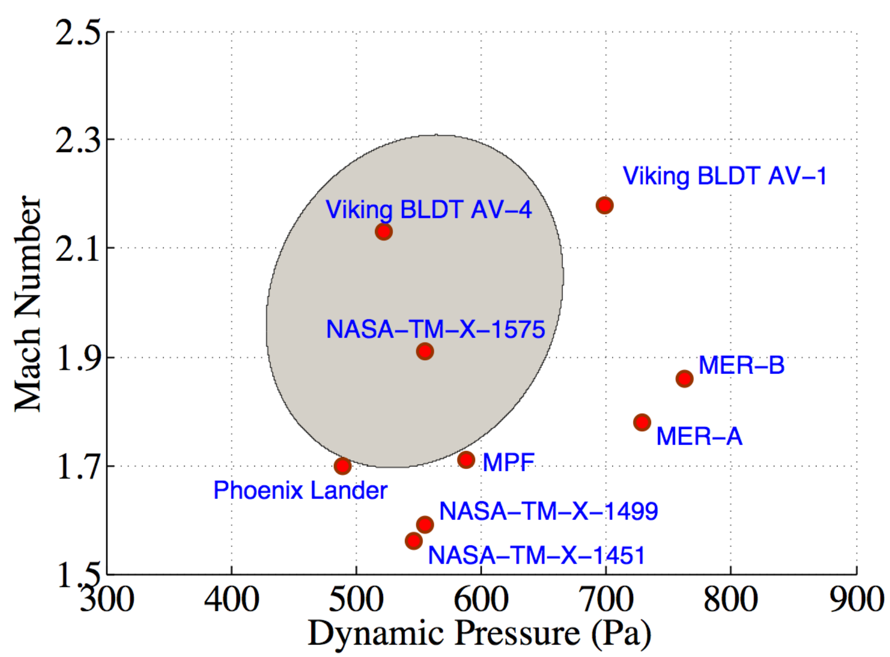


Predict heating rate uncertainty

Filtering Problem



Supersonic parachute

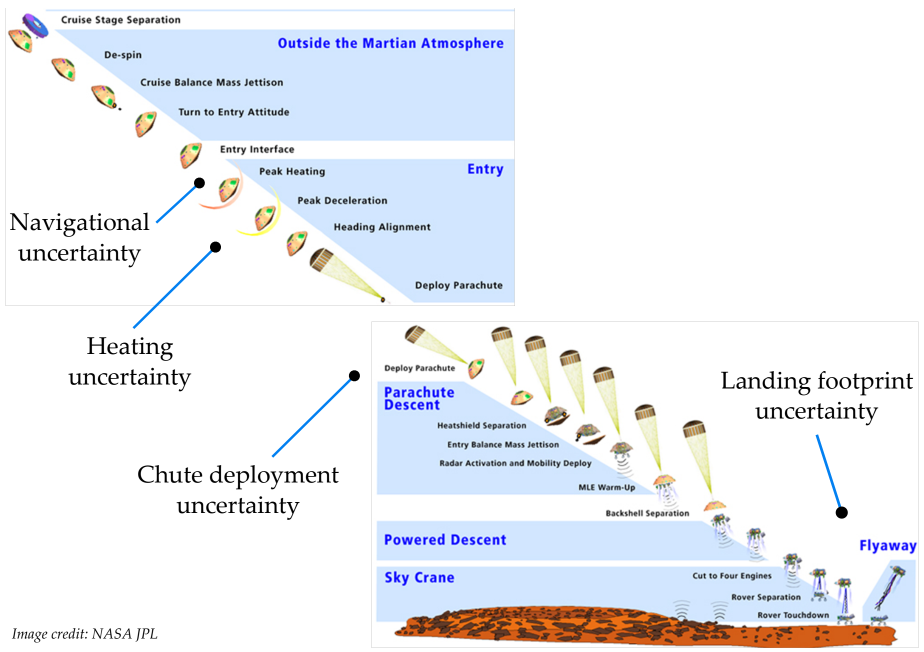


Estimate state to deploy parachute

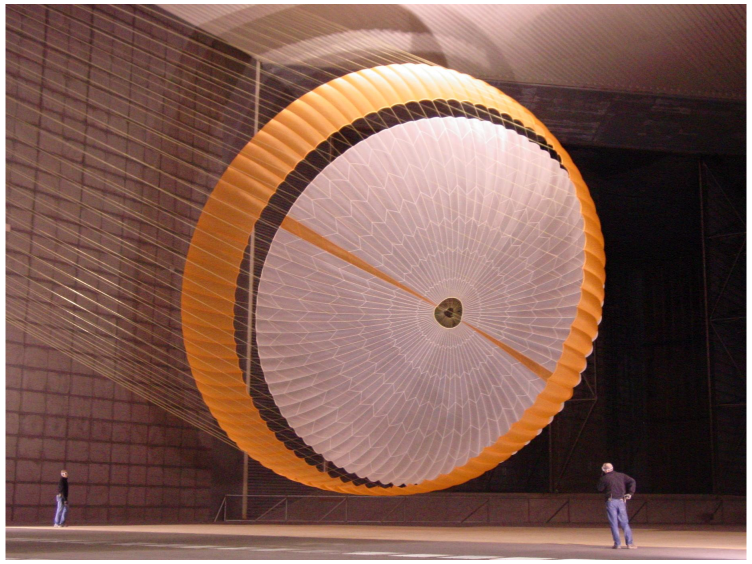
Control Problem

PDFs in Mars Entry-Descent-Landing

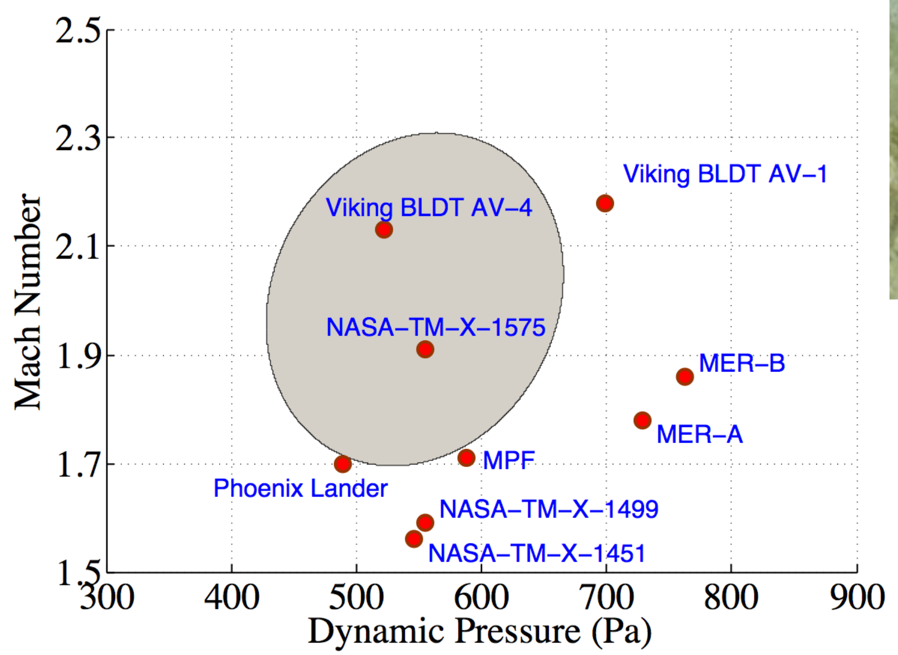
Prediction Problem



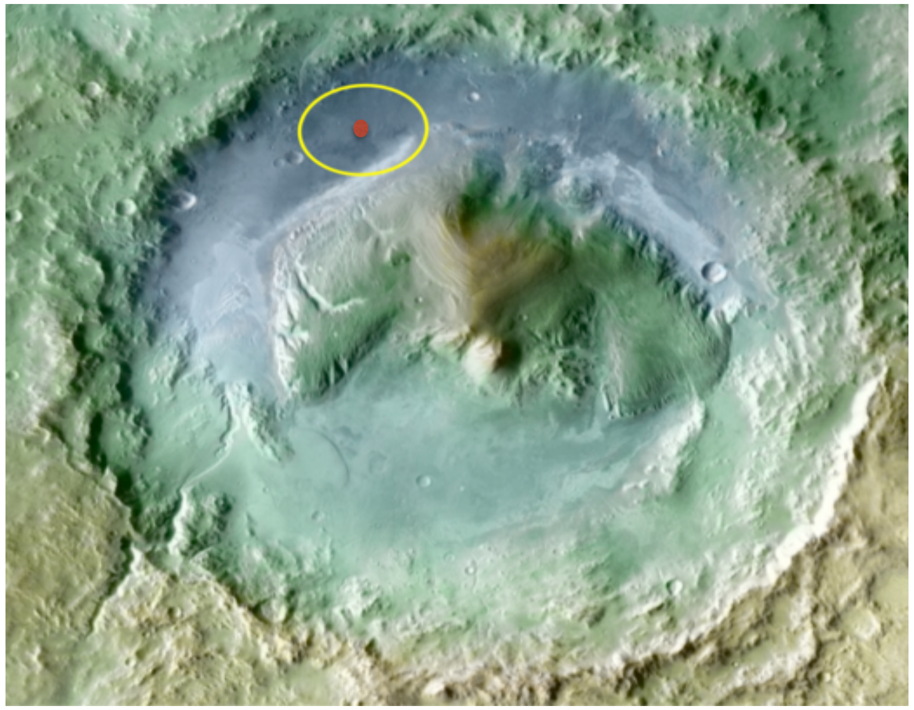
Filtering Problem



Supersonic parachute



Control Problem



Gale Crater (4.49S, 137.42E)

Predict heating rate uncertainty

Estimate state to deploy parachute

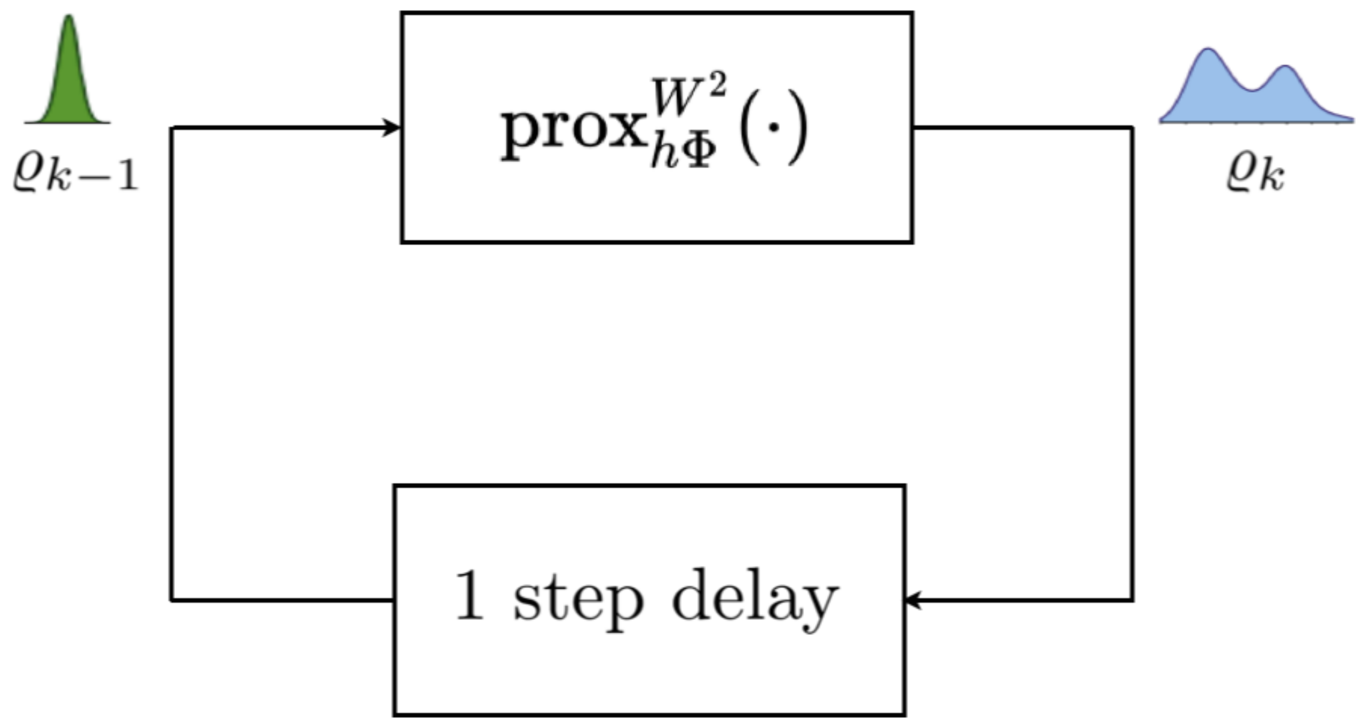
Steer state PDF to achieve desired landing footprint accuracy

Solving prediction problem as gradient flow

What's New?

Main idea: Solve $\dfrac{\partial \rho}{\partial t} = \mathcal{L}_{\rm FP}\rho, \;\; \rho(x, t=0) = \rho_0$ **as gradient flow in** $\mathcal{P}_2(\mathcal{X})$

Infinite dimensional variational recursion:



Proximal operator: $\varrho_k = \mathrm{prox}_{h\Phi}^{W^2}(\varrho_{k-1}) := \underset{\varrho \in \mathcal{P}_2(\mathcal{X})}{\arg\inf} \left\{ \dfrac{1}{2} W^2(\varrho, \varrho_{k-1}) + h\Phi(\varrho) \right\}$

Optimal transport cost: $W^2(\varrho, \varrho_{k-1}) := \inf\limits_{\pi \in \Pi(\varrho, \varrho_{k-1})} \displaystyle\int_{\mathcal{X}\times\mathcal{X}} c(x,y)\, \mathrm{d}\pi(x,y)$

Free energy functional: $\Phi(\varrho) := \displaystyle\int_{\mathcal{X}} \psi \varrho \, \mathrm{d}x + \beta^{-1} \int_{\mathcal{X}} \varrho \log \varrho \, \mathrm{d}x$

Gradient Flow

Gradient Flow in $\mathcal{X}$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\nabla\varphi(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

Recursion:

$$\begin{aligned}\boldsymbol{x}_k &= \boldsymbol{x}_{k-1} - h\nabla\varphi(\boldsymbol{x}_k) \\ &= \underset{\boldsymbol{x}\in\mathcal{X}}{\arg\min}\left\{\frac{1}{2}\|\boldsymbol{x}-\boldsymbol{x}_{k-1}\|_2^2 + h\varphi(\boldsymbol{x})\right\} \\ &=: \mathrm{prox}_{h\varphi}^{\|\cdot\|_2}(\boldsymbol{x}_{k-1})\end{aligned}$$

Convergence:

$$\boldsymbol{x}_k \to \boldsymbol{x}(t = kh) \quad \text{as} \quad h \downarrow 0$$

Gradient Flow in $\mathcal{P}_2(\mathcal{X})$

$$\frac{\partial\rho}{\partial t} = -\nabla^W\Phi(\rho), \quad \rho(\boldsymbol{x}, 0) = \rho_0$$

Recursion:

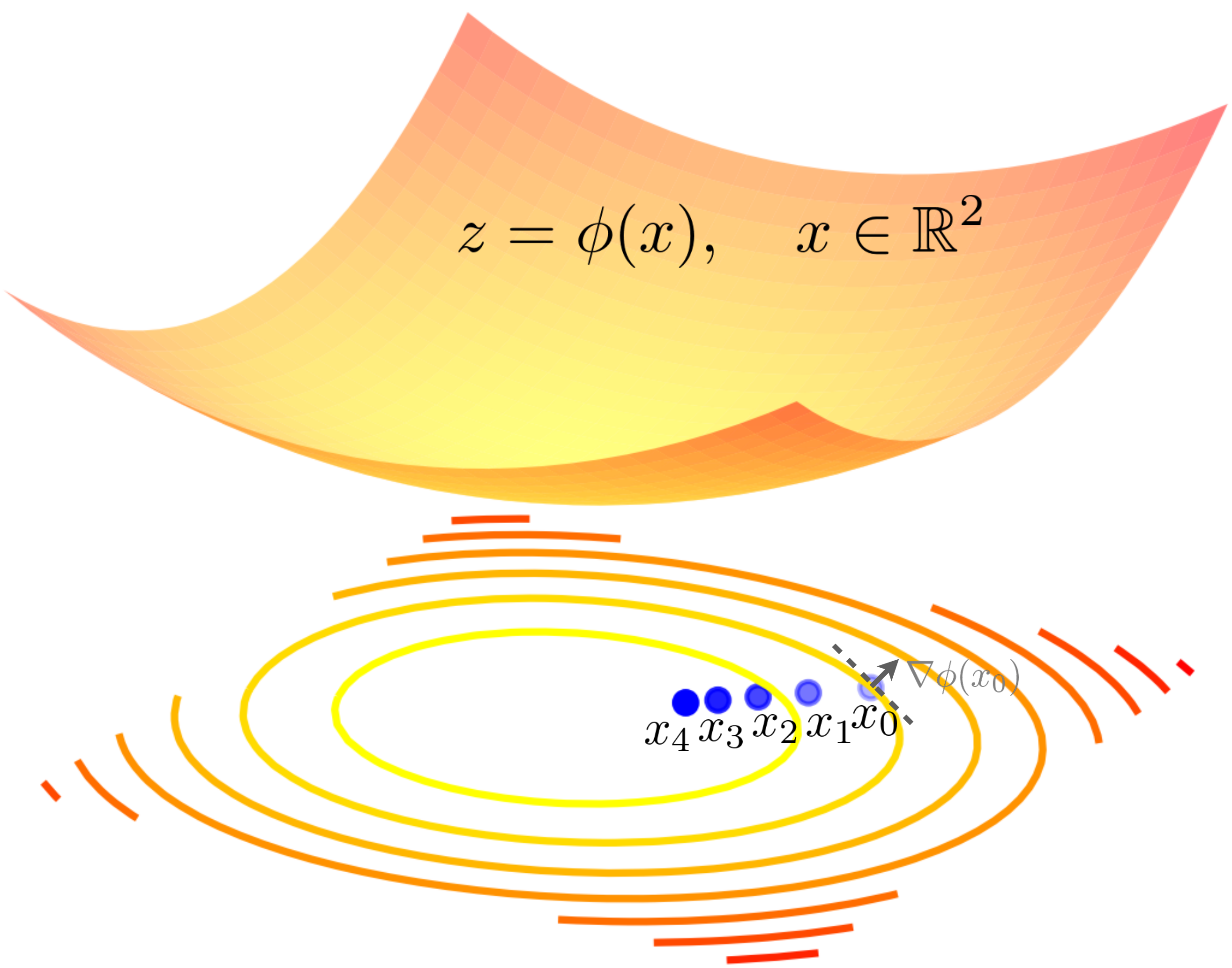
$$\begin{aligned}\rho_k &= \rho(\cdot, t = kh) \\ &= \underset{\rho\in\mathcal{P}_2(\mathcal{X})}{\arg\min}\left\{\frac{1}{2}W^2(\rho, \rho_{k-1}) + h\Phi(\rho)\right\} \\ &=: \mathrm{prox}_{h\Phi}^{W^2}(\rho_{k-1})\end{aligned}$$

Convergence:

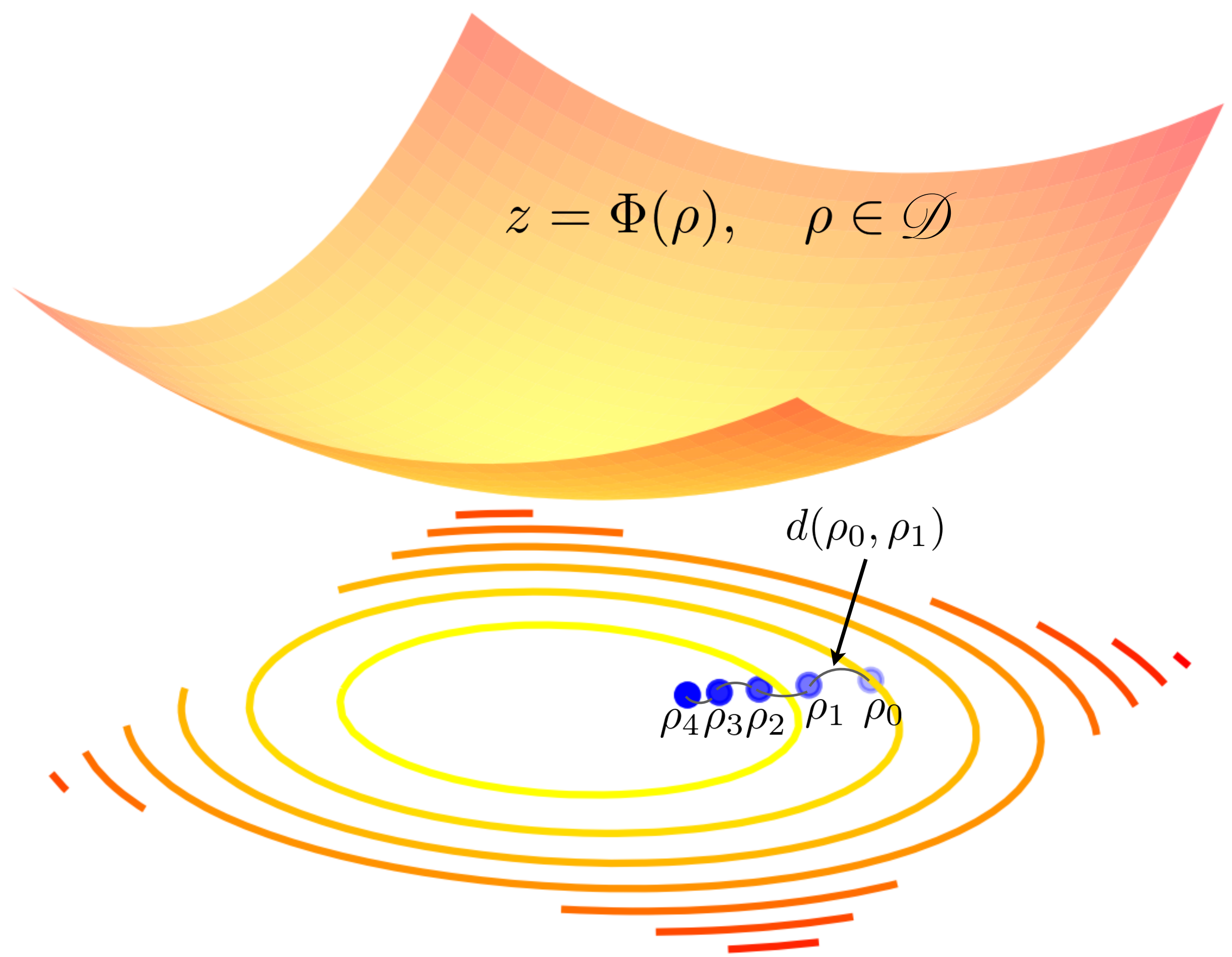
$$\rho_k \to \rho(\cdot, t = kh) \quad \text{as} \quad h \downarrow 0$$

Gradient Flow

Gradient Flow in $\mathcal{X}$

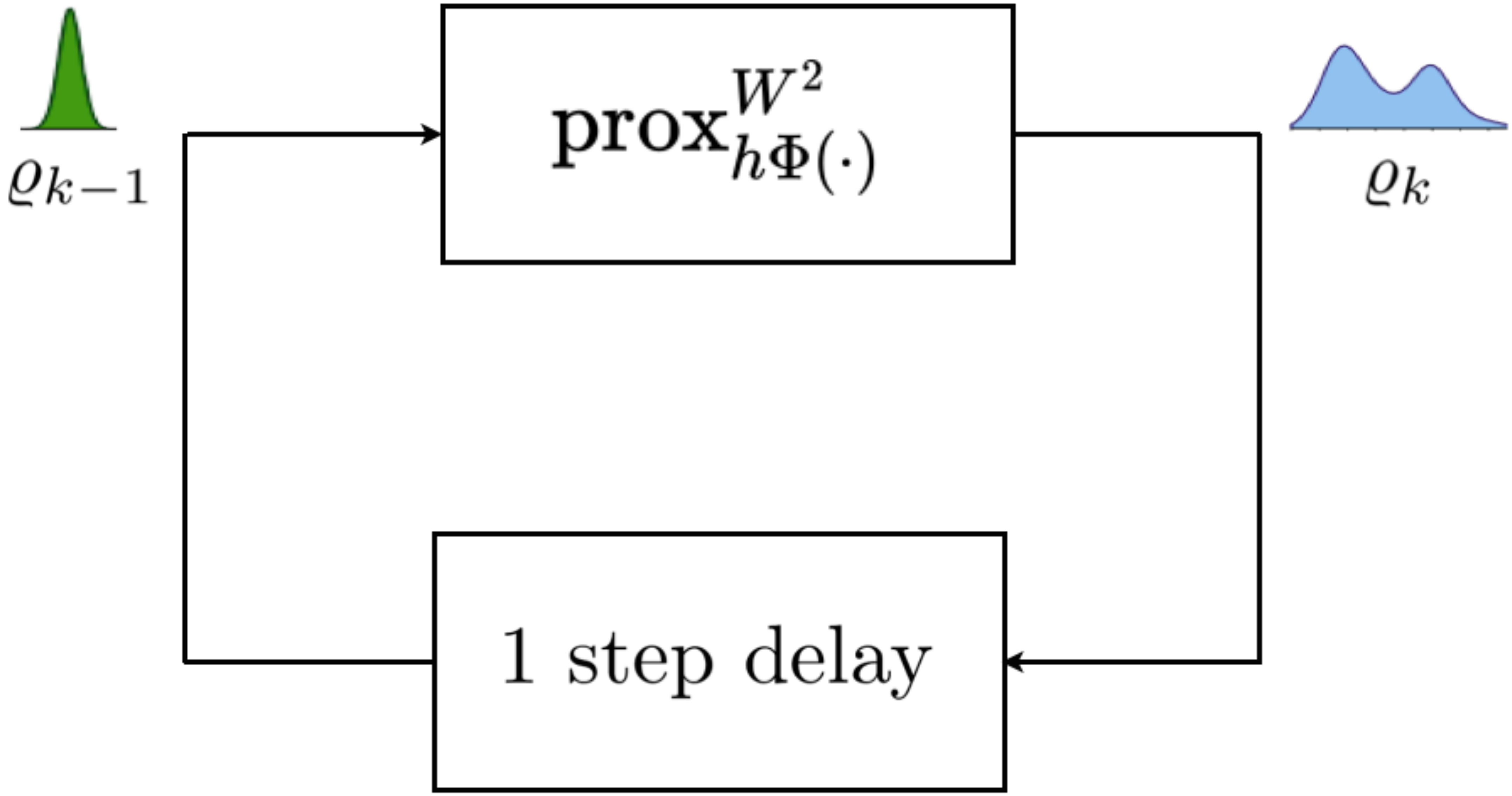


Gradient Flow in $\mathcal{P}_2(\mathcal{X})$



Algorithm: Gradient Ascent on the Dual Space

Uncertainty propagation via point clouds



No spatial discretization or function approximation

Algorithm: Gradient Ascent on the Dual Space

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \psi \rho) + \beta^{-1} \Delta \rho$$

$\Updownarrow$ **Proximal Recursion**

$$\rho_k = \rho(\boldsymbol{x}, t = kh) = \underset{\rho \in \mathcal{P}_2(\mathbb{R}^n)}{\arg\inf} \left\{ \frac{1}{2} W^2(\rho, \rho_{k-1}) + h\, \Phi(\rho) \right\}$$

Algorithm: Gradient Ascent on the Dual Space

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \psi \rho) + \beta^{-1} \Delta \rho$$

$\Updownarrow$ **Proximal Recursion**

$$\rho_k = \rho(\boldsymbol{x}, t = kh) = \underset{\rho \in \mathcal{P}_2(\mathbb{R}^n)}{\arg\inf} \left\{ \frac{1}{2} W^2(\rho, \rho_{k-1}) + h\, \Phi(\rho) \right\}$$

$\Downarrow$ **Discrete Primal Formulation**

$$\boldsymbol{\varrho}_k = \underset{\boldsymbol{\varrho}}{\arg\min} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + h \, \langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \rangle \right\}$$

Algorithm: Gradient Ascent on the Dual Space

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \psi \rho) + \beta^{-1} \Delta \rho$$

$\Updownarrow$ **Proximal Recursion**

$$\rho_k = \rho(\boldsymbol{x}, t = kh) = \underset{\rho \in \mathcal{P}_2(\mathbb{R}^n)}{\arg\inf} \left\{ \frac{1}{2} W^2(\rho, \rho_{k-1}) + h\,\Phi(\rho) \right\}$$

$\Downarrow$ **Discrete Primal Formulation**

$$\boldsymbol{\varrho}_k = \arg\min_{\boldsymbol{\varrho}} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + h \left\langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \right\rangle \right\}$$

$\Downarrow$ **Entropic Regularization**

$$\boldsymbol{\varrho}_k = \arg\min_{\boldsymbol{\varrho}} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + \epsilon H(\boldsymbol{M}) + h \left\langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \right\rangle \right\}$$

Algorithm: Gradient Ascent on the Dual Space

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \psi \rho) + \beta^{-1} \Delta \rho$$

$\Updownarrow$ **Proximal Recursion**

$$\rho_k = \rho(\boldsymbol{x}, t = kh) = \underset{\rho \in \mathcal{P}_2(\mathbb{R}^n)}{\arg\inf} \left\{ \frac{1}{2} W^2(\rho, \rho_{k-1}) + h\, \Phi(\rho) \right\}$$

$\Downarrow$ **Discrete Primal Formulation**

$$\boldsymbol{\varrho}_k = \arg\min_{\boldsymbol{\varrho}} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + h \left\langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \right\rangle \right\}$$

$\Downarrow$ **Entropic Regularization**

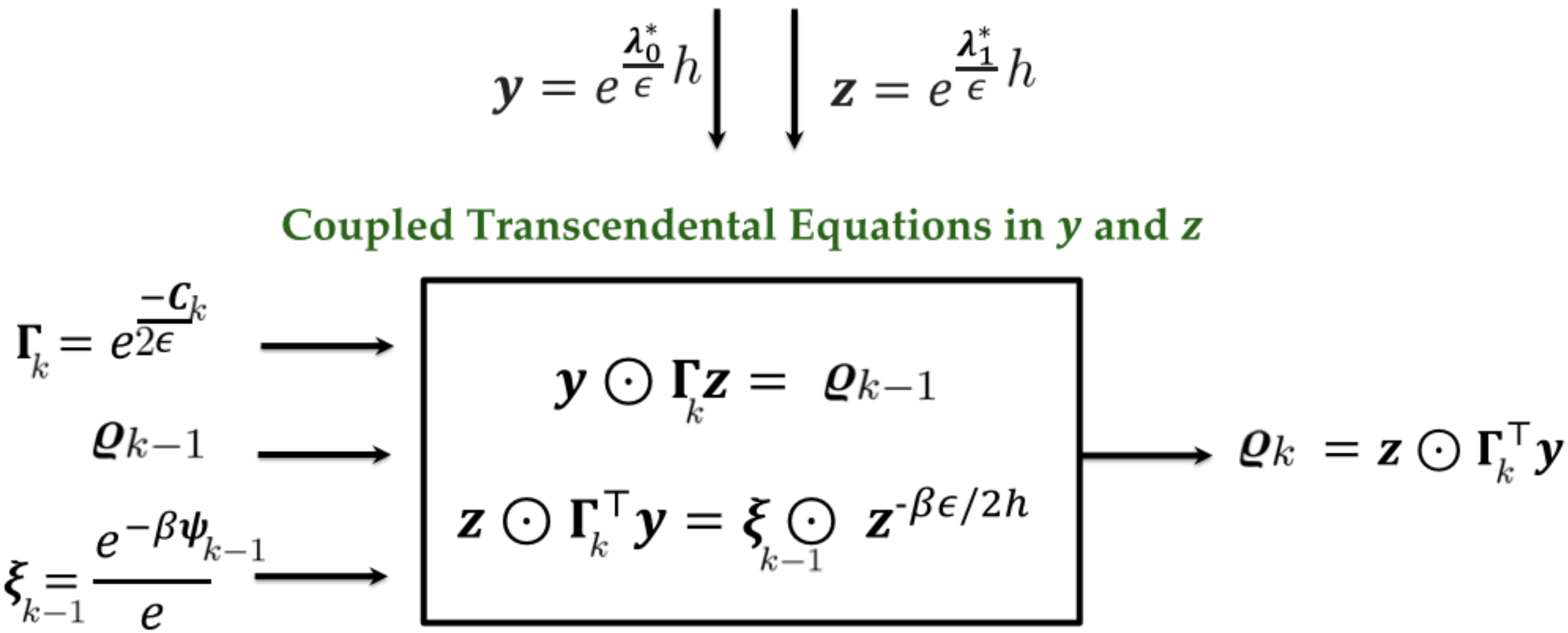
$$\boldsymbol{\varrho}_k = \arg\min_{\boldsymbol{\varrho}} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + \epsilon H(\boldsymbol{M}) + h \left\langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \right\rangle \right\}$$

$\Updownarrow$ **Dualization**

$$\boldsymbol{\lambda}_0^{\text{opt}}, \boldsymbol{\lambda}_1^{\text{opt}} = \arg\max_{\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1 \geq 0} \Big\{ \langle \boldsymbol{\lambda}_0, \boldsymbol{\varrho}_{k-1} \rangle - F^{\star}(-\boldsymbol{\lambda}_1)$$

$$- \frac{\epsilon}{h} \left(\exp(\boldsymbol{\lambda}_0^{\top} h / \epsilon) \exp(-\boldsymbol{C}_k / 2\epsilon) \exp(\boldsymbol{\lambda}_1 h / \epsilon) \right) \Big\}$$

Fixed Point Recursion

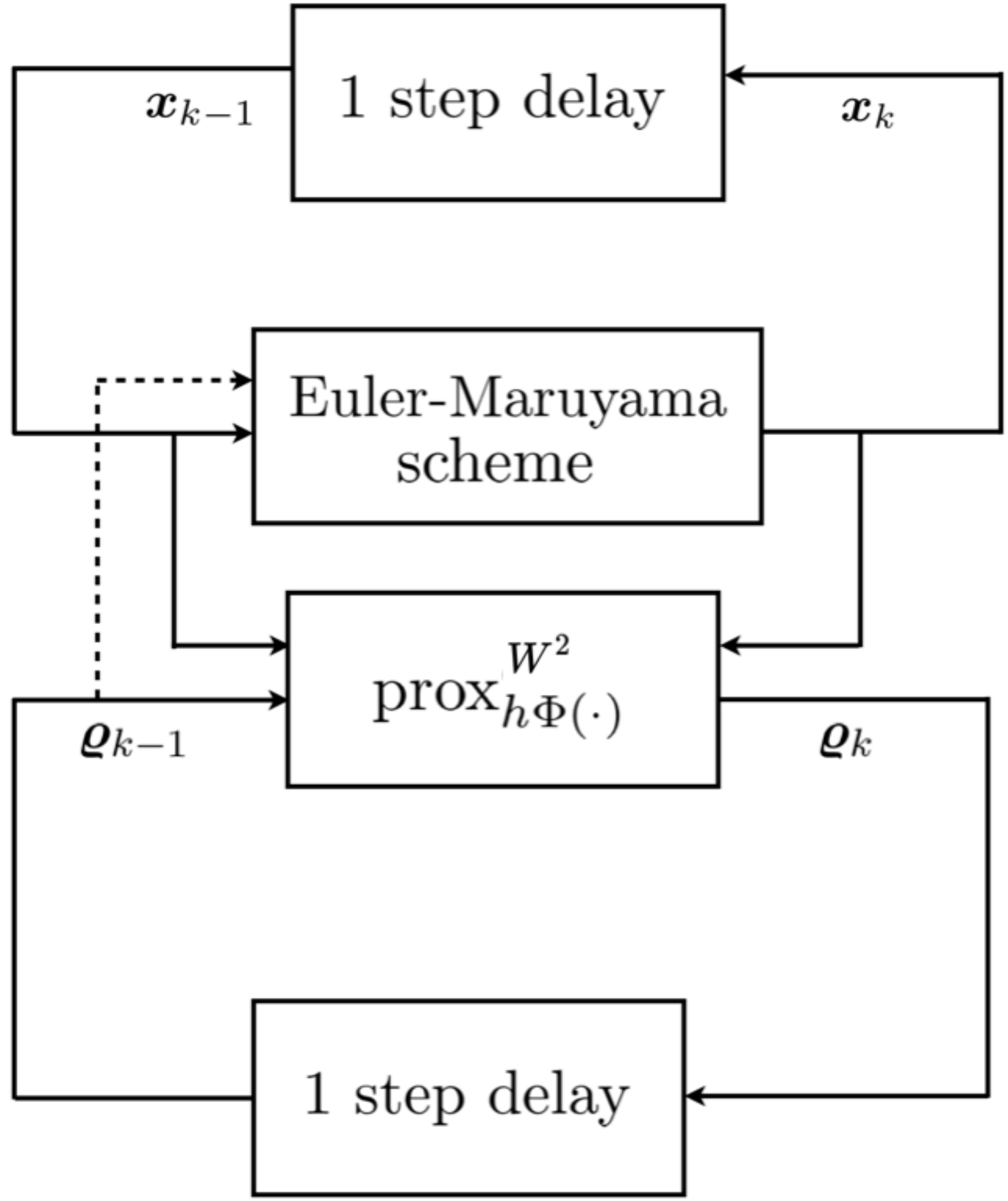


Theorem: Consider the recursion on the cone $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0}$

$$\boldsymbol{y} \odot (\boldsymbol{\Gamma}_k \boldsymbol{z}) = \varrho_{k-1}, \quad \boldsymbol{z} \odot \left(\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}\right) = \boldsymbol{\xi}_{k-1} \odot \boldsymbol{z}^{-\frac{\beta\epsilon}{h}},$$

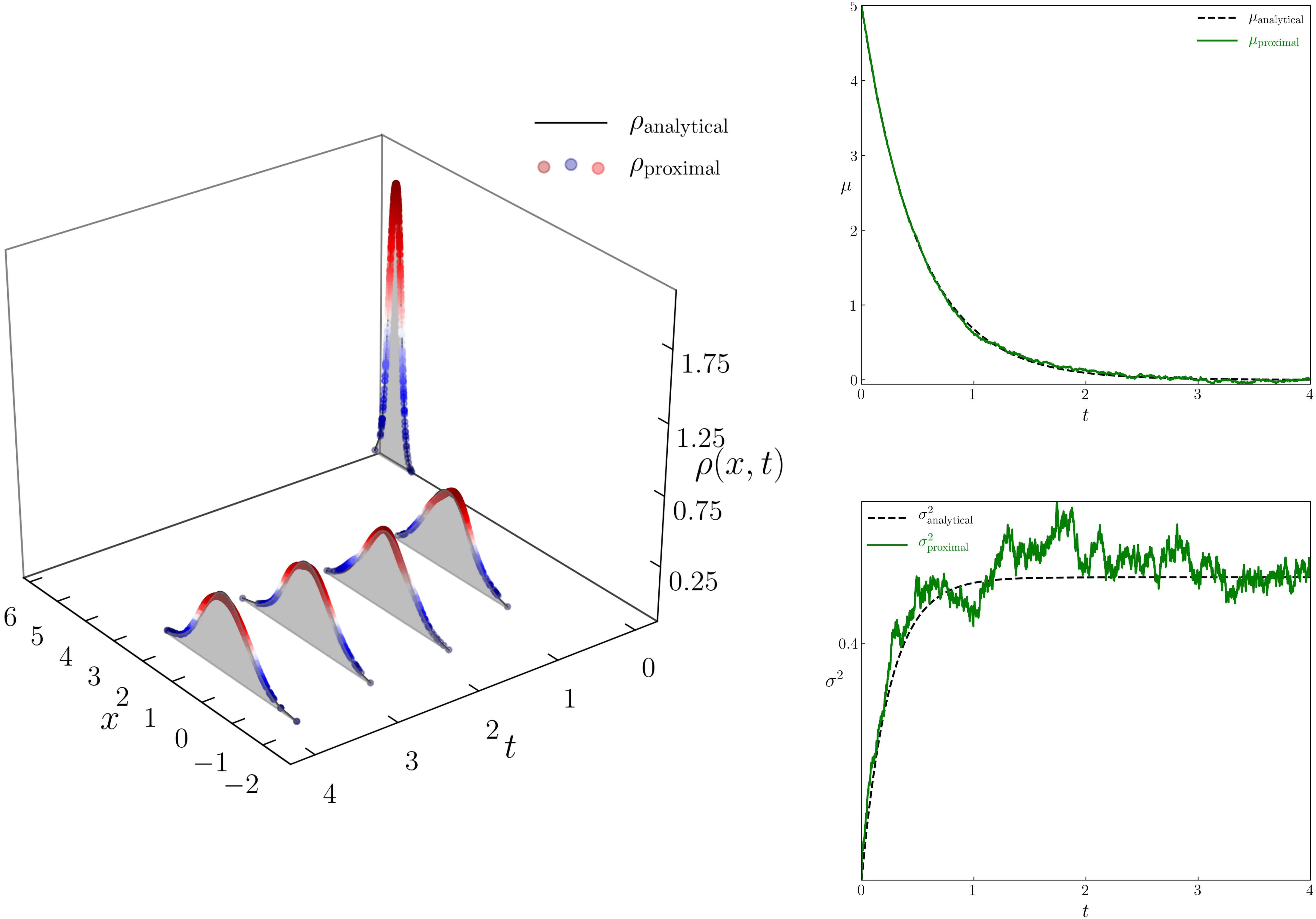
Then the solution $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ gives the proximal update $\varrho_k = \boldsymbol{z}^* \odot (\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}^*)$

Algorithmic Setup

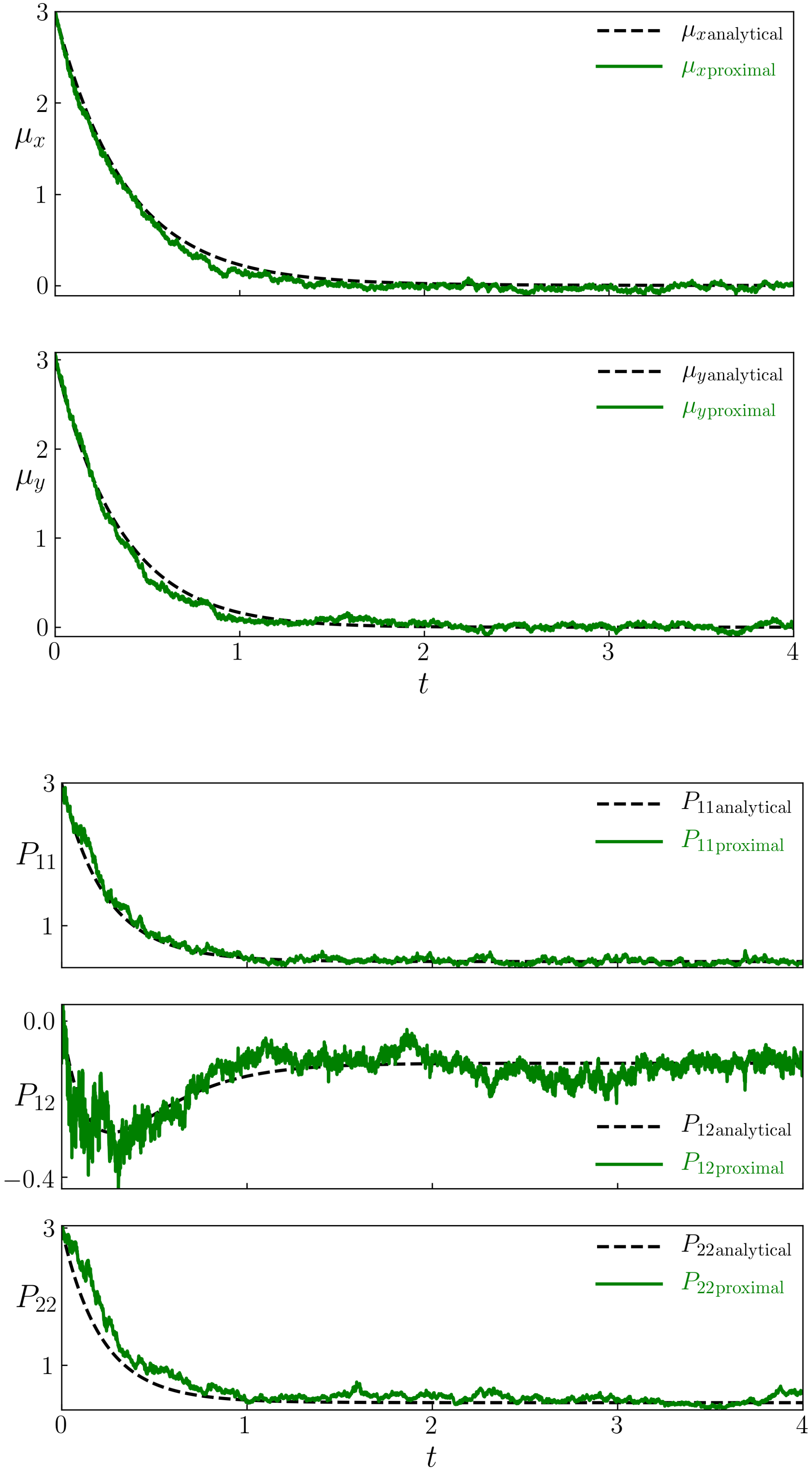


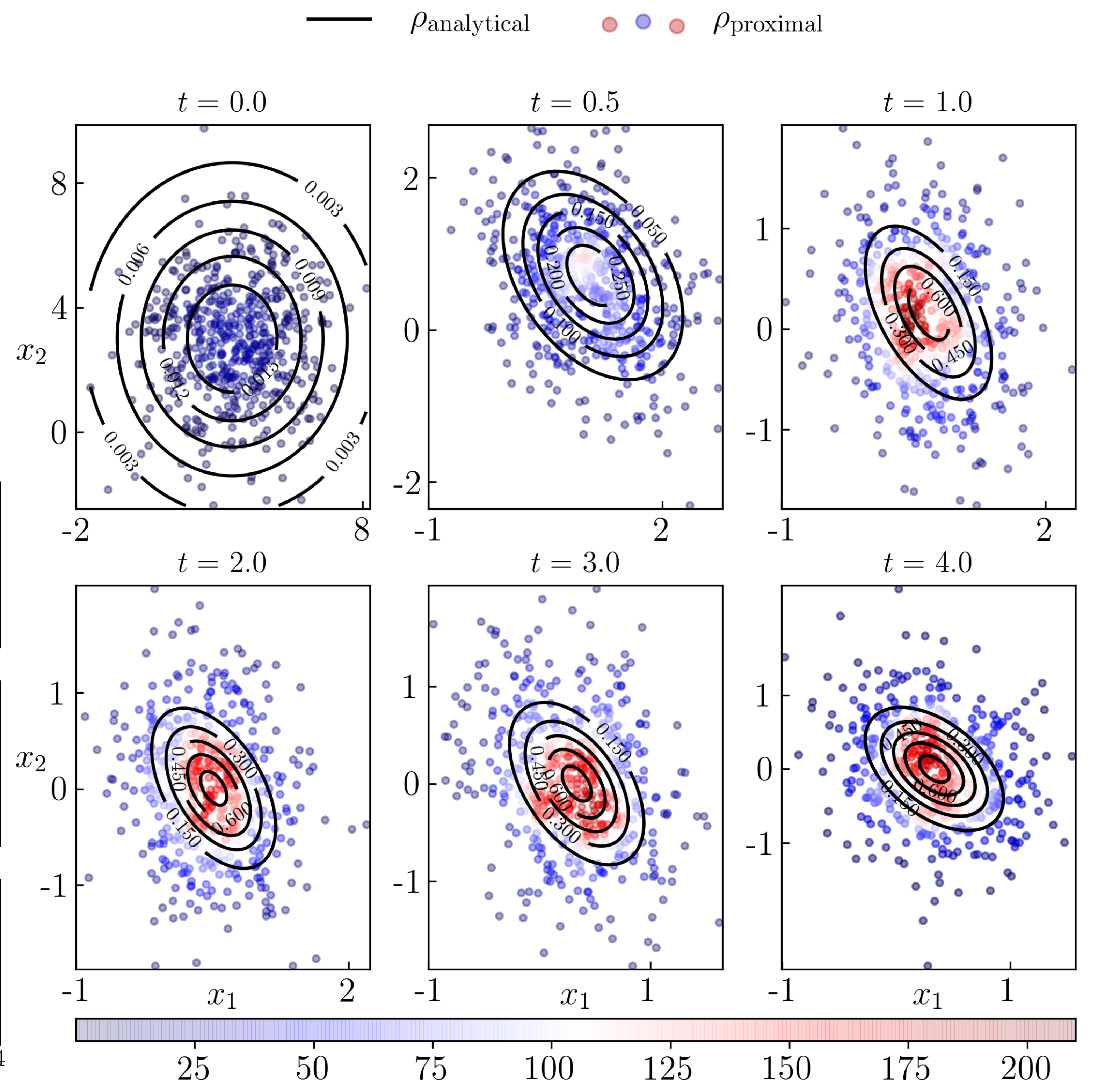
Theorem: Block co-ordinate iteration of $(\mathbf{y}, \mathbf{z})$ recursion is contractive on $\mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0}$.

Proximal Prediction: 1D Linear Gaussian

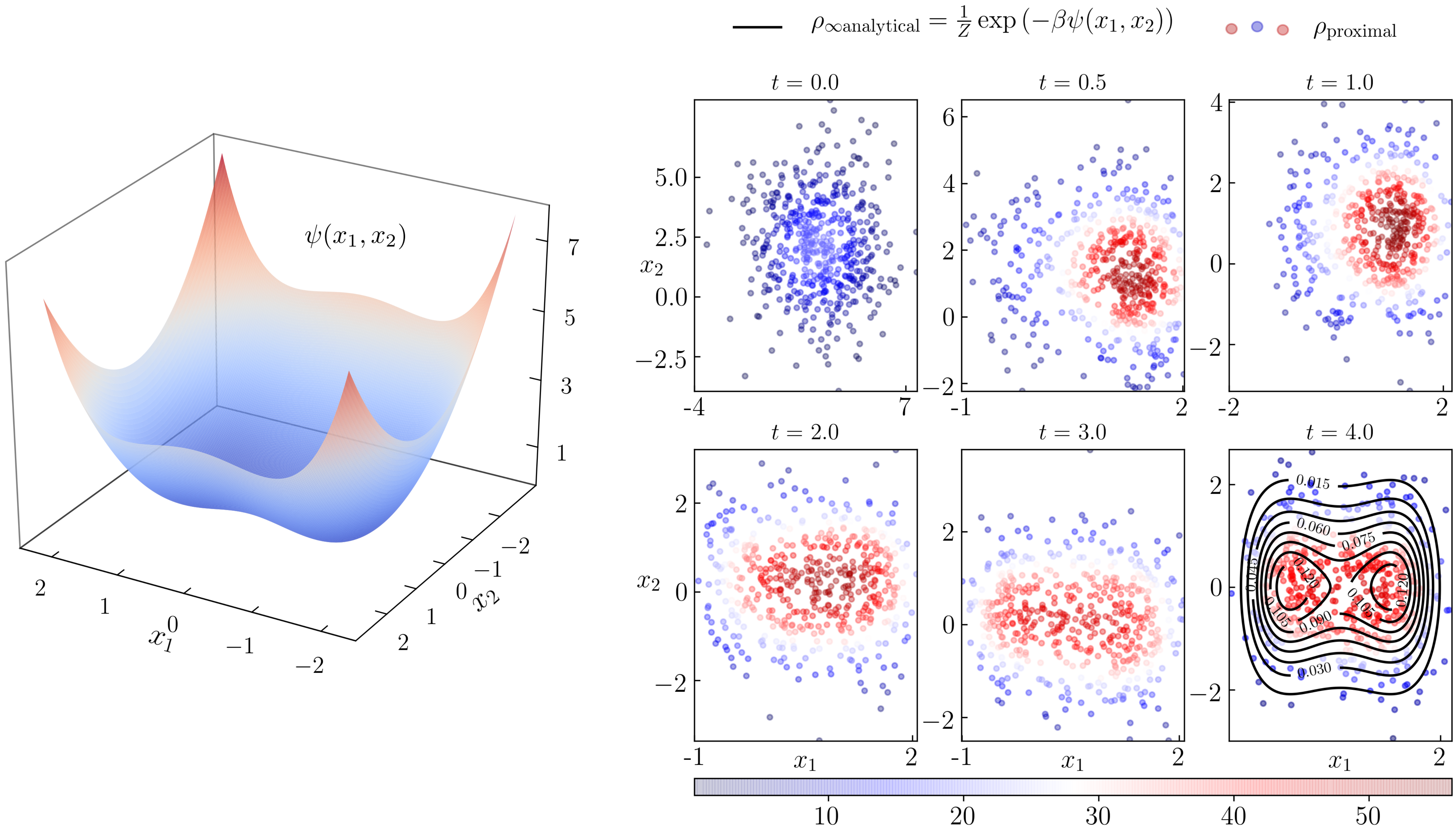


Proximal Prediction: 2D Linear Gaussian

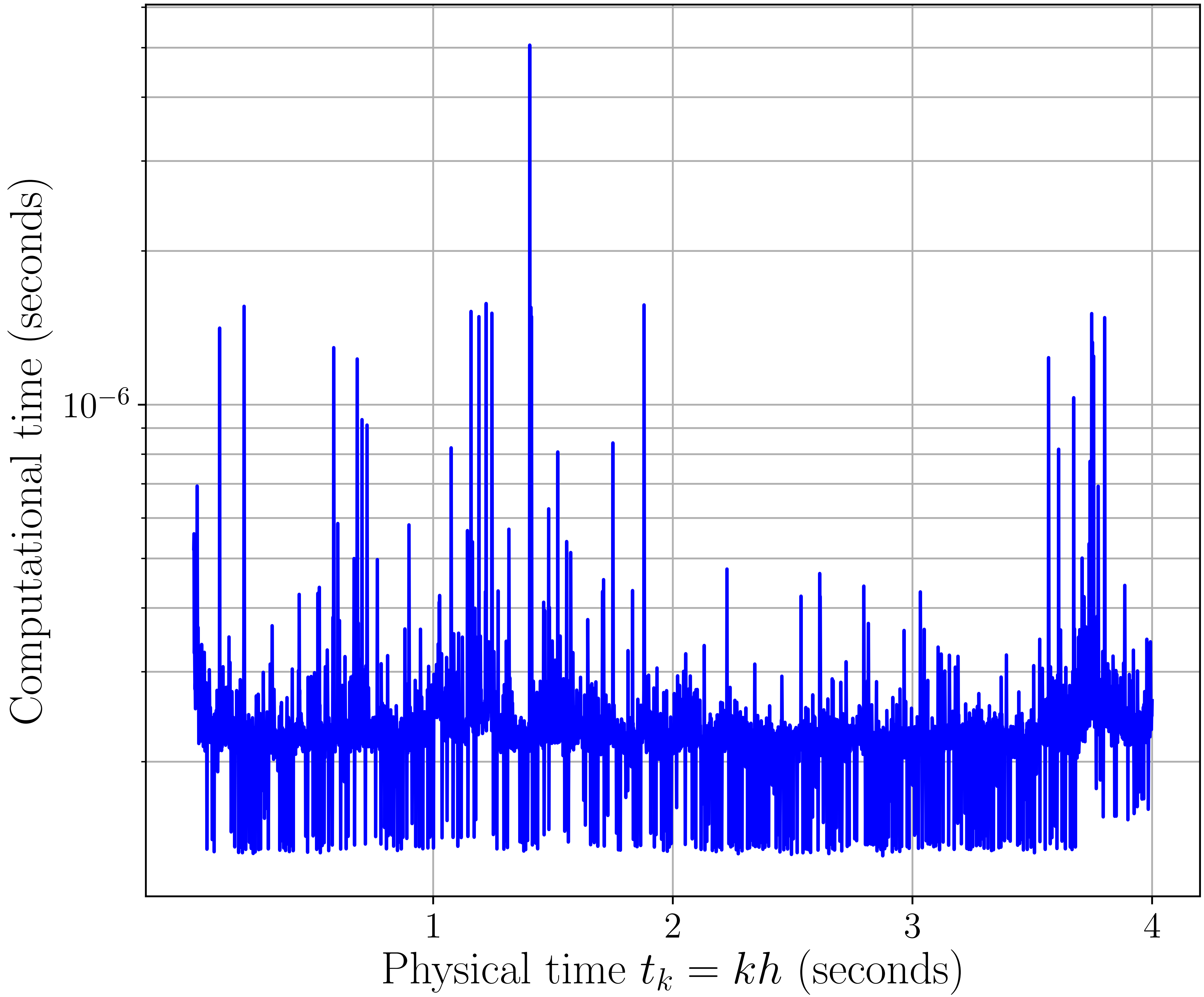




Proximal Prediction: 2D Nonlinear Non-Gaussian



Computational Time: 2D Nonlinear Non-Gaussian



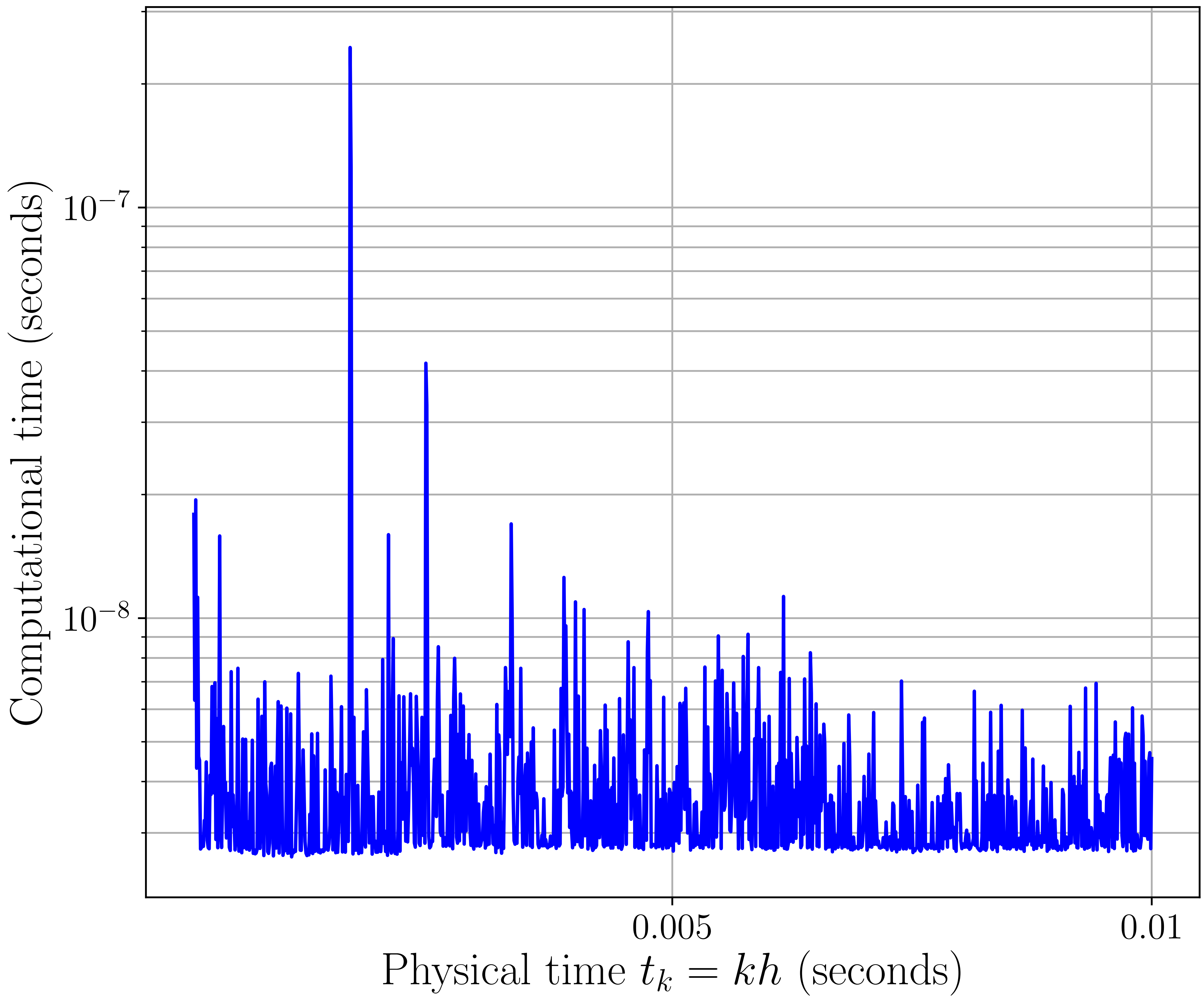
Proximal Prediction: Satellite in Geocentric Orbit

Here, $\mathcal{X} \equiv \mathbb{R}^6$

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \\ \mathrm{d}v_x \\ \mathrm{d}v_y \\ \mathrm{d}v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ -\dfrac{\mu x}{r^3} + (f_x)_{\text{pert}} - \gamma v_x \\ -\dfrac{\mu y}{r^3} + (f_y)_{\text{pert}} - \gamma v_y \\ -\dfrac{\mu z}{r^3} + (f_z)_{\text{pert}} - \gamma v_z \end{pmatrix} \mathrm{d}t + \sqrt{2\beta^{-1}\gamma} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{d}w_1 \\ \mathrm{d}w_2 \\ \mathrm{d}w_3 \end{pmatrix},$$

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}_{\text{pert}} = \begin{pmatrix} s\theta\, c\phi & c\theta\, c\phi & -s\phi \\ s\theta\, s\phi & c\theta\, s\phi & c\phi \\ c\theta & -s\theta & 0 \end{pmatrix} \begin{pmatrix} \dfrac{k}{2r^4}\left(3(s\theta)^2 - 1\right) \\ -\dfrac{k}{r^5} s\theta\, c\theta \\ 0 \end{pmatrix}, k := 3J_2 R_{\mathrm{E}}^2, \mu = \text{constant}$$

Computational Time: Satellite in Geocentric Orbit



Extensions: Nonlocal interactions

PDF dependent sample path dynamics:

$$\mathrm{d}\mathbf{x} = -\left(\nabla U\left(\mathbf{x}\right) + \nabla \rho * V\right)\mathrm{d}t + \sqrt{2\beta^{-1}}\,\mathrm{d}\mathbf{w}$$

Mckean-Vlasov-Fokker-Planck-Kolmogorov integro PDE:

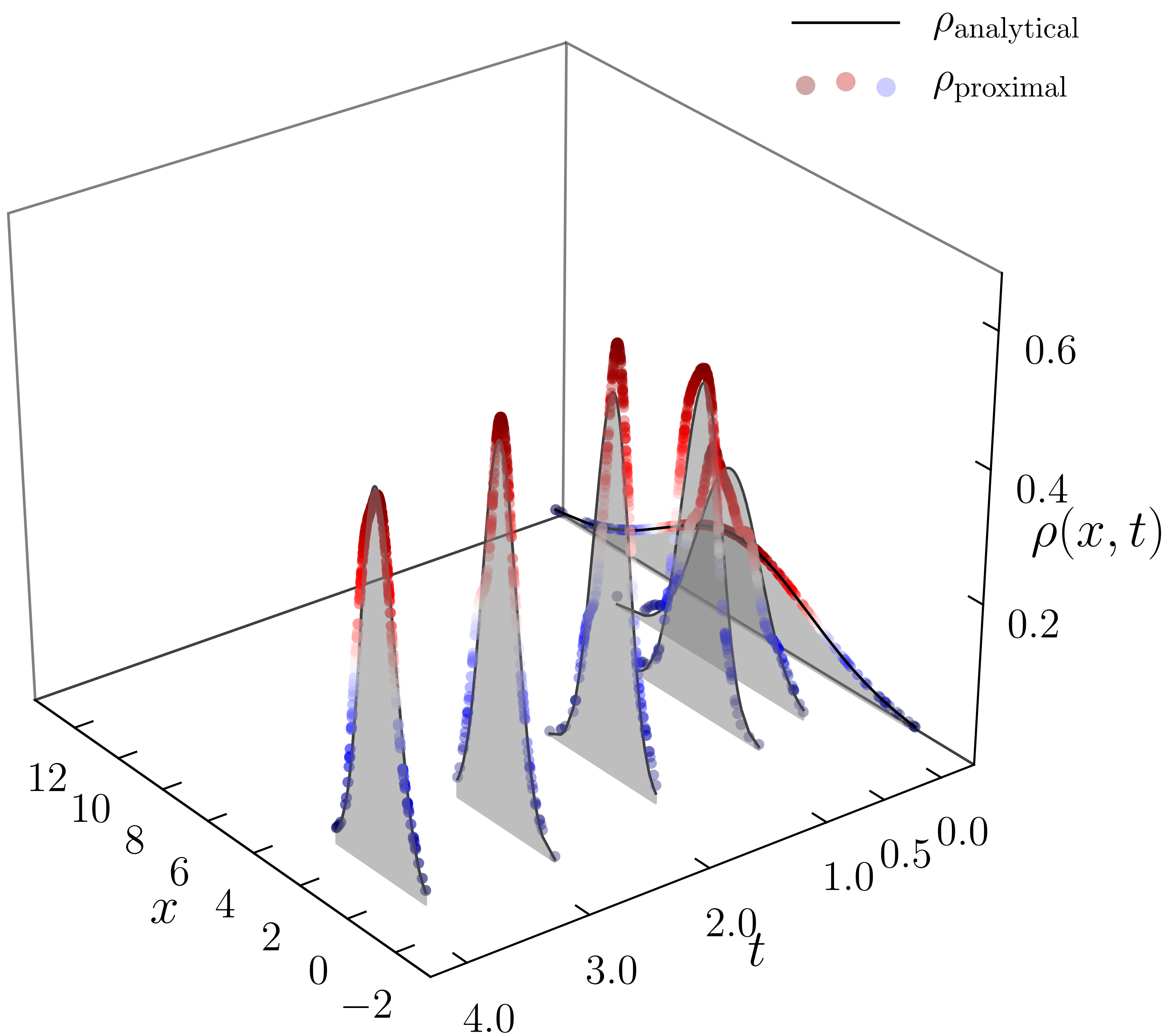
$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \nabla \left(U + \rho * V\right)\right) + \beta^{-1} \Delta \rho$$

Free energy:

$$F(\rho) := \mathbb{E}_{\rho}\left[U + \beta^{-1} \rho \log \rho + \rho * V\right]$$

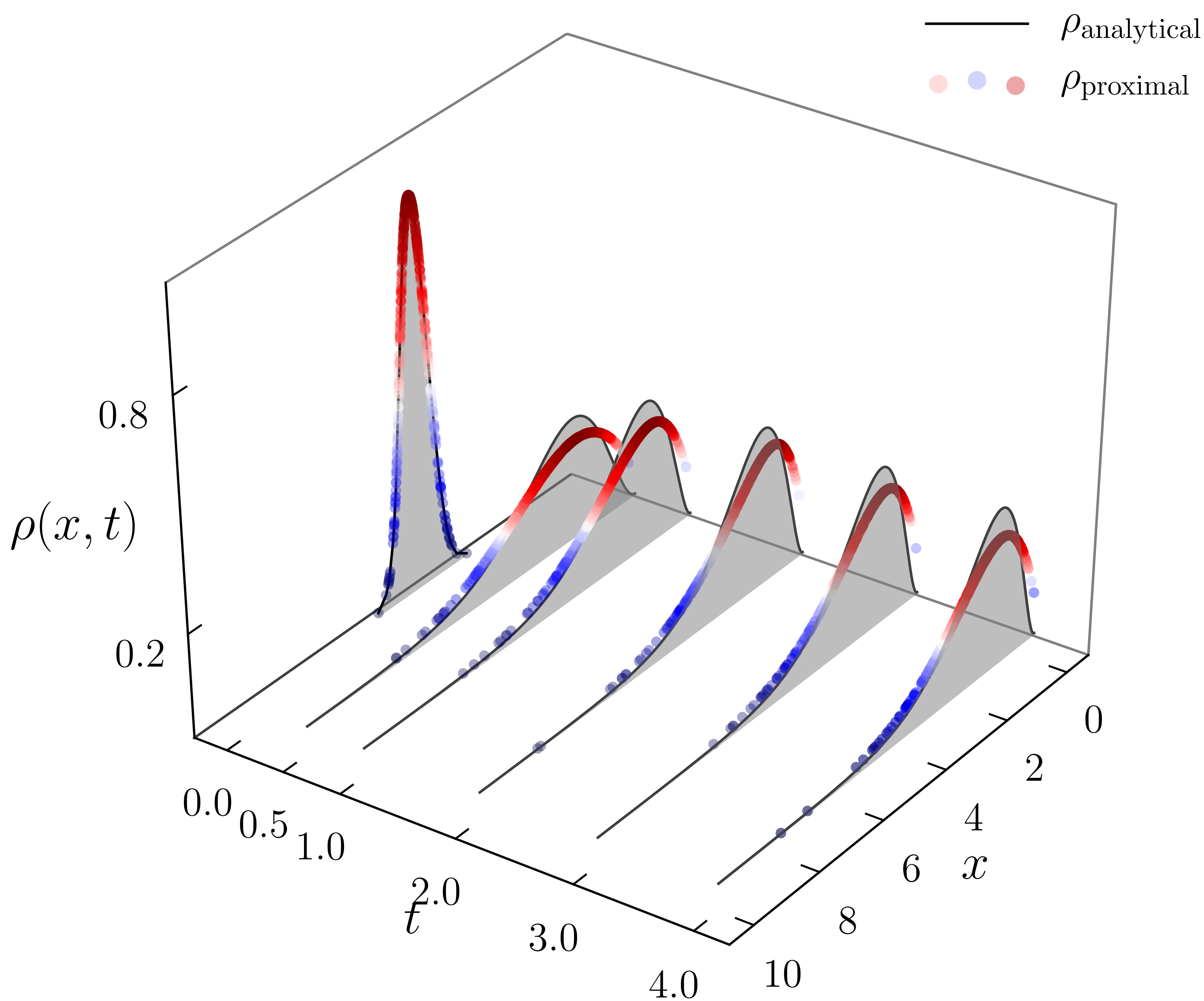
Extensions: Nonlocal interactions (contd.)

$$U(\cdot) = V(\cdot) = \|\cdot\|_2^2$$



Extensions: Multiplicative Noise

Cox-Ingersoll-Ross: $\mathrm{d}x = a\,(\theta - x)\;\mathrm{d}t + b\sqrt{x}\,\mathrm{d}w$, $2a > b^2, \theta > 0$



Details on Proximal Prediction

— K.F. Caluya, and A.H., Proximal Recursion for Solving the Fokker-Planck Equation, ACC 2019.

— K.F. Caluya, and A.H., Gradient Flow Algorithms for Density Propagation in Stochastic Systems, under review in TAC.

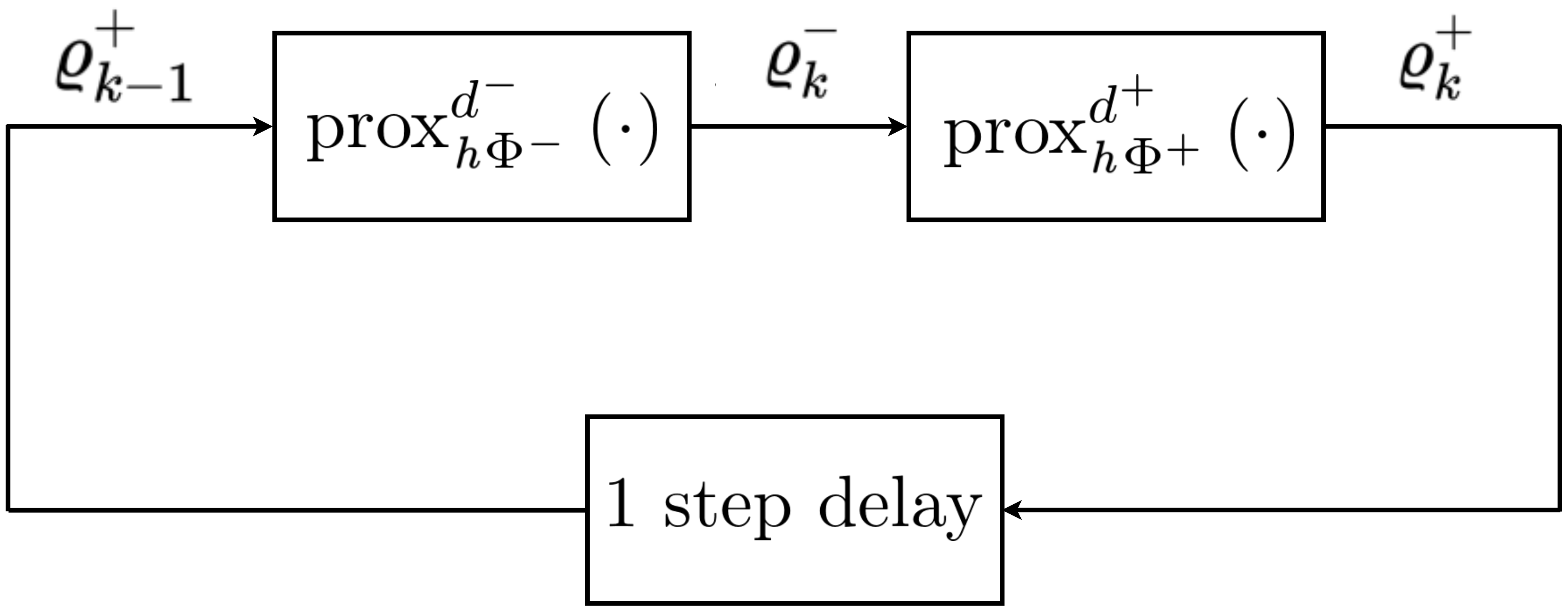
Solving filtering problem as gradient flow

What's New?

Main idea: Solve the Kushner-Stratonovich SPDE

$\mathrm{d}\rho^+ = \left[\mathcal{L}_{\mathrm{FP}}\mathrm{d}t + \mathcal{L}\left(\mathrm{d}z, \mathrm{d}t, \rho^+\right)\right]\rho^+, \; \rho(x, t=0) = \rho_0$ as gradient flow in $\mathcal{P}_2(\mathcal{X})$

Recursion of {deterministic ∘ stochastic} proximal operators:



Convergence: $\varrho_k^+(h) \to \rho^+(x, t = kh) \quad \text{as} \quad h \downarrow 0$

For prior, as before: $d^- \equiv W^2, \quad \Phi^- \equiv \mathbb{E}_{\varrho}\left[\psi + \beta^{-1}\log\varrho\right]$

For posterior: $d^+ \equiv d_{\mathrm{FR}}^2$ or $D_{\mathrm{KL}}, \quad \Phi^+ \equiv \dfrac{1}{2}\mathbb{E}_{\varrho^+}\left[(y_k - h(x))^\top R^{-1}(y_k - h(x))\right]$

Explicit Recovery of Kalman-Bucy Filter

Model:

$$\mathrm{d}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)\mathrm{d}t + \mathbf{B}\mathrm{d}\mathbf{w}(t), \qquad \mathrm{d}\mathbf{w}(t) \sim \mathcal{N}(0, \mathbf{Q}\mathrm{d}t)$$

$$\mathrm{d}\mathbf{z}(t) = \mathbf{C}\mathbf{x}(t)\mathrm{d}t + \mathrm{d}\mathbf{v}(t), \qquad \mathrm{d}\mathbf{v}(t) \sim \mathcal{N}(0, \mathbf{R}\mathrm{d}t)$$

Given $\mathbf{x}(0) \sim \mathcal{N}(\mu_0, \mathbf{P}_0)$, want to recover:

$\mathbf{P}^{+}\mathbf{C}\mathbf{R}^{-1}$

$$\mathrm{d}\mu^{+}(t) = \mathbf{A}\mu^{+}(t)\mathrm{d}t + \mathbf{K}(t) \left(\mathrm{d}\mathbf{z}(t) - \mathbf{C}\mu^{+}(t)\mathrm{d}t\right),$$

$$\dot{\mathbf{P}}^{+}(t) = \mathbf{A}\mathbf{P}^{+}(t) + \mathbf{P}^{+}(t)\mathbf{A}^{\top} + \mathbf{B}\mathbf{Q}\mathbf{B}^{\top} - \mathbf{K}(t)\mathbf{R}\mathbf{K}(t)^{\top}.$$

— A.H. and T.T. Georgiou, Gradient Flows in Uncertainty Propagation and Filtering of Linear Gaussian Systems, CDC 2017.

— A.H. and T.T. Georgiou, Gradient Flows in Filtering and Fisher-Rao Geometry, ACC 2018.

Explicit Recovery of Wonham Filter

Model:

$$x(t) \sim \text{Markov}(Q),$$
$$\mathrm{d}z(t) = h(x(t))\,\mathrm{d}t \;+\; \sigma_v(t)\mathrm{d}v(t)$$

State space: $\Omega := \{a_1, \ldots, a_m\}$

Posterior $\pi^+(t) := \{\pi_1^+(t), \ldots, \pi_m^+(t)\}$ **solves the nonlinear SDE:**

$$\mathrm{d}\pi^+(t) = \pi^+(t)Q\,\mathrm{d}t \;+\; \frac{1}{(\sigma_v(t))^2}\pi^+(t)\Big(H - \widehat{h}(t)I\Big)\Big(\mathrm{d}z(t) - \widehat{h}(t)\mathrm{d}t\Big),$$

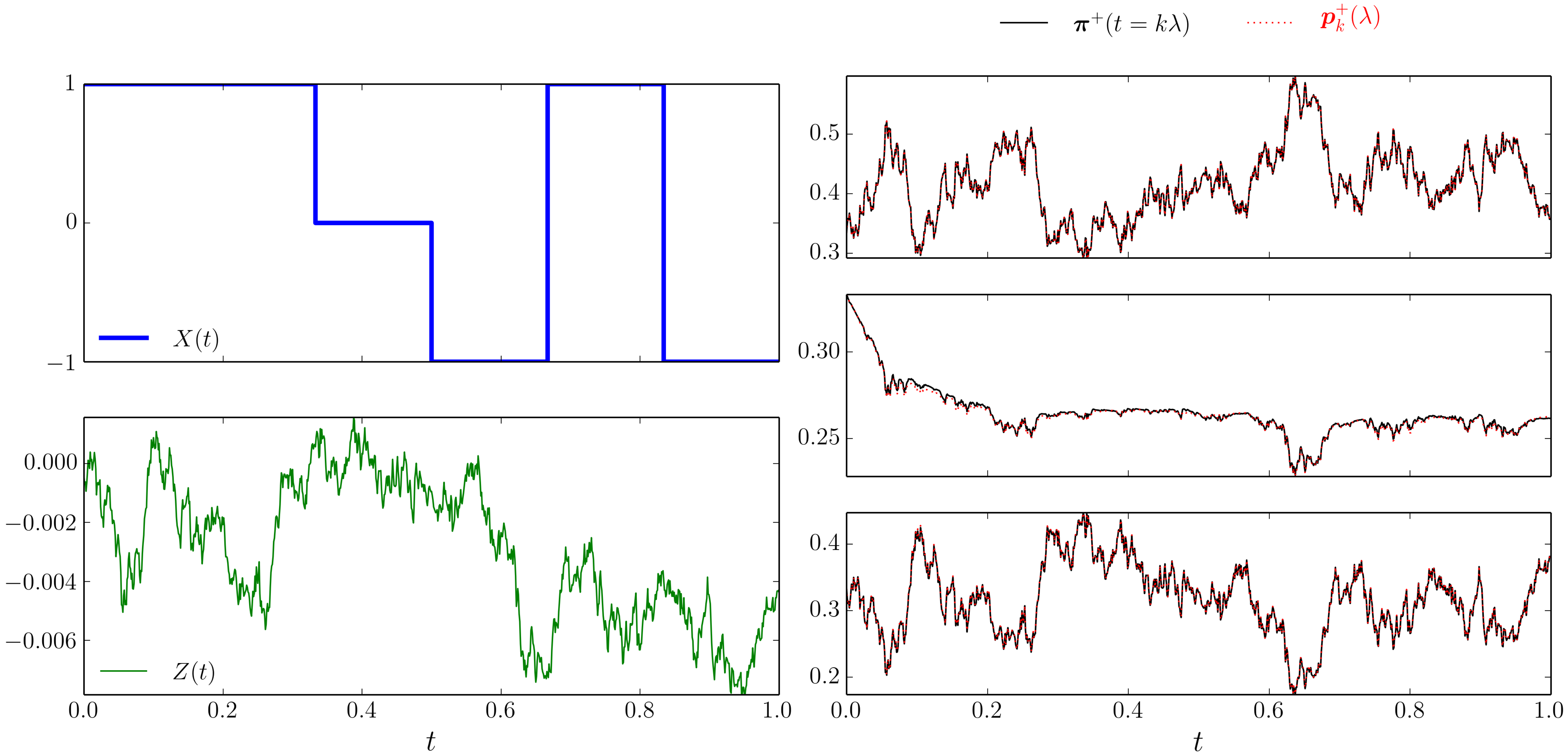
where $H := \text{diag}(h(a_1), \ldots, h(a_m)), \quad \widehat{h}(t) := \sum_{i=1}^{m} h(a_i)\pi_i^+(t),$

Initial condition: $\pi^+(t=0) = \pi_0,$

By defn. $\pi^+(t) = \mathbb{P}(x(t) = a_i \mid z(s), 0 \le s \le t)$

— A.H. and T.T. Georgiou, Proximal Recursion for the Wonham Filter, CDC 2019, under review.

Numerical Results for Wonham Filter



— A.H. and T.T. Georgiou, Proximal Recursion for the Wonham Filter, CDC 2019, under review.

Solving density steering using gradient flow

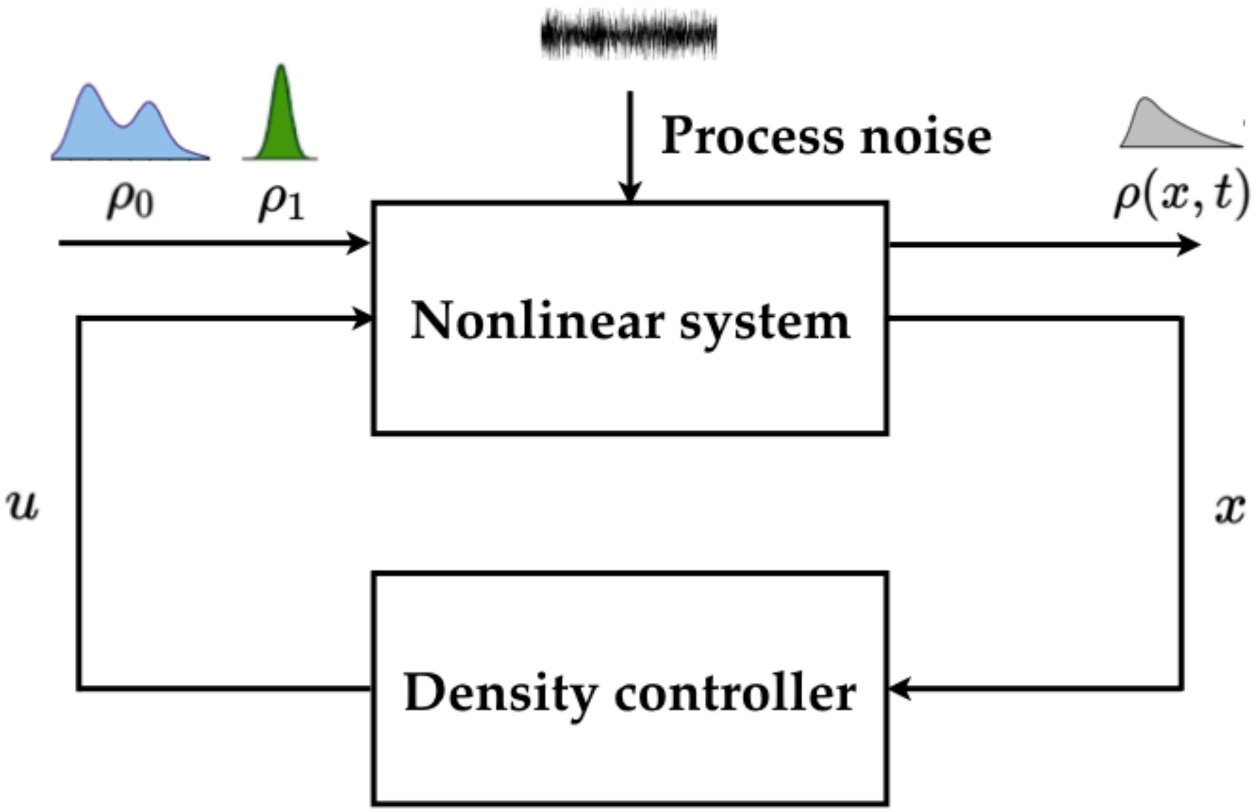
Finite Horizon Feedback Density Steering

$$\underset{u\in\mathcal{U}}{\text{minimize}}\ \mathbb{E}\left[\int_0^1 \|u\|_2^2\, \mathrm{d}t\right]$$

subject to

$$\mathrm{d}x = f(x,u,t)\,\mathrm{d}t \;+\; g(x,t)\,\mathrm{d}w,$$

$$x(t=0)\sim\rho_0,\quad x(t=1)\sim\rho_1$$



Consider simple case: $f(x,u,t)\equiv f(x,t)+u,\quad g=\sqrt{2\epsilon}$

Coupled Nonlinear PDE system (Fokker-Planck + Hamilton-Jacobi-Bellman):

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho(f+\nabla\psi)) + \epsilon\Delta\rho,$$

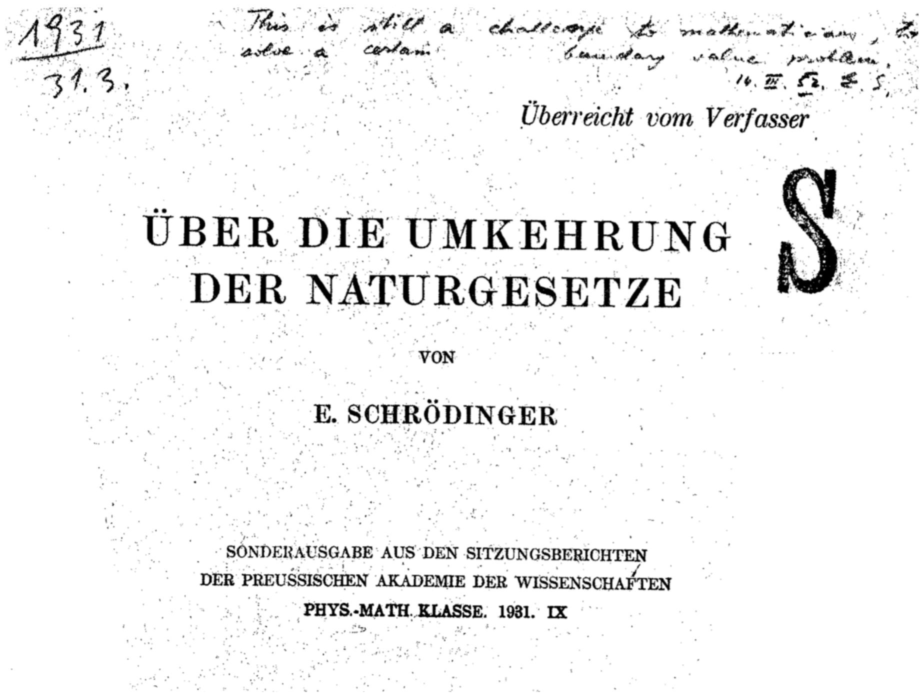
$$\frac{\partial\psi}{\partial t} = -\langle f,\nabla\psi\rangle - \frac{\|\nabla\psi\|_2^2}{2} - \epsilon\Delta\psi.$$

LTV case is solved (boundary coupled system of Riccati ODEs):

— Y. Chen, T.T. Georgiou, and M. Pavon, Optimal Transport Over a Linear Dynamical System, TAC 2017 [George S. Axelby Outstanding Paper Award]

Solution via Schrödinger Bridge

Schrödinger's (until recently) forgotten papers:


1931
31.3.

This is still a challenge to mathematicians, to solve a certain boundary value problem.

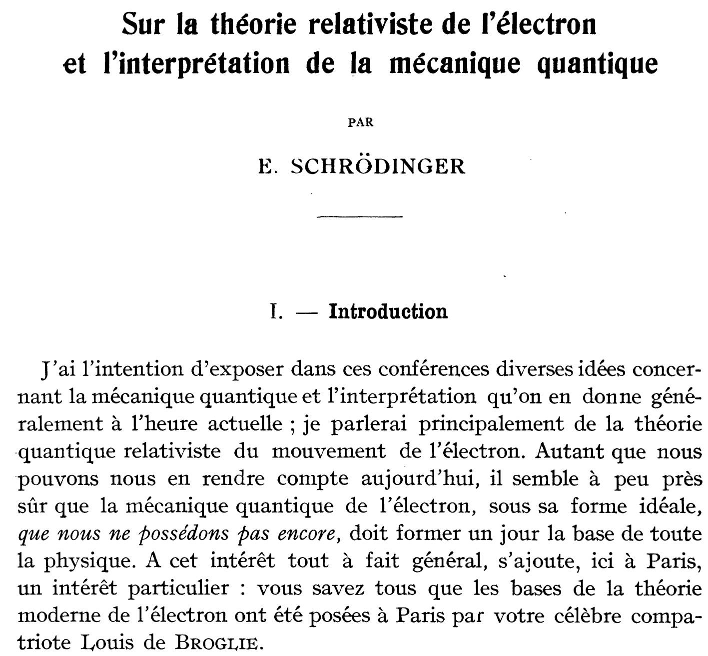
Überreicht vom Verfasser

ÜBER DIE UMKEHRUNG
DER NATURGESETZE

VON

E. SCHRÖDINGER

SONDERAUSGABE AUS DEN SITZUNGSBERICHTEN
DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN
PHYS.-MATH. KLASSE. 1931. IX


Sur la théorie relativiste de l'électron
et l'interprétation de la mécanique quantique

PAR

E. SCHRÖDINGER

I. — **Introduction**

J'ai l'intention d'exposer dans ces conférences diverses idées concernant la mécanique quantique et l'interprétation qu'on en donne généralement à l'heure actuelle ; je parlerai principalement de la théorie quantique relativiste du mouvement de l'électron. Autant que nous pouvons nous en rendre compte aujourd'hui, il semble à peu près sûr que la mécanique quantique de l'électron, sous sa forme idéale, *que nous ne possédons pas encore*, doit former un jour la base de toute la physique. A cet intérêt tout à fait général, s'ajoute, ici à Paris, un intérêt particulier : vous savez tous que les bases de la théorie moderne de l'électron ont été posées à Paris par votre célèbre compatriote Louis de Broglie.

Schrödinger's contribution:

2 coupled nonlinear PDEs ⟶ boundary-coupled linear PDEs!!

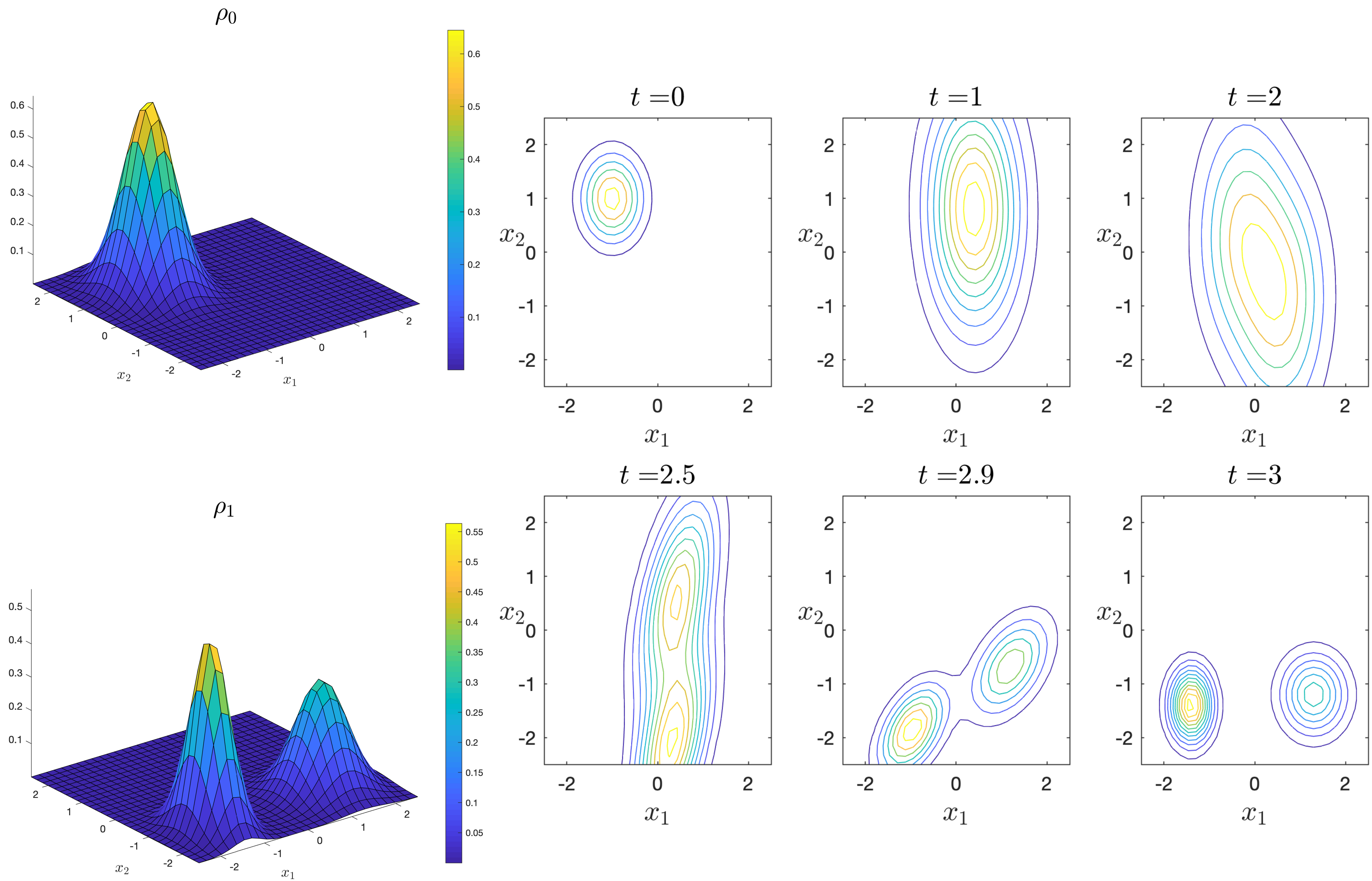
For $f = -\nabla U$:

$$\frac{\partial \widehat{\varphi}}{\partial t} = \nabla \cdot (\widehat{\varphi}\, \nabla U) + \epsilon \Delta \widehat{\varphi}, \quad \widehat{\varphi}(x, t=0) = \widehat{\varphi}_0,$$

$$\frac{\partial \varphi}{\partial t} = \nabla U \cdot \nabla \varphi - \epsilon \Delta \varphi, \widehat{\varphi}(x, t=1) = \varphi_1,$$

Optimal controlled joint state PDF: $\rho^*(x,t) = \widehat{\varphi}(x,t)\varphi(x,t)$

Feedback Density Steering: Proximal Algorithms



Details on Feedback Density Control for Nonlinear Systems

— K.F. Caluya, and A.H., Finite Horizon Density Control for Static State Feedback Linearizable Systems, under review in TAC.

— K.F. Caluya, W. Li, and A.H., Schrodinger Bridge with Nonlinear Drift, working draft.

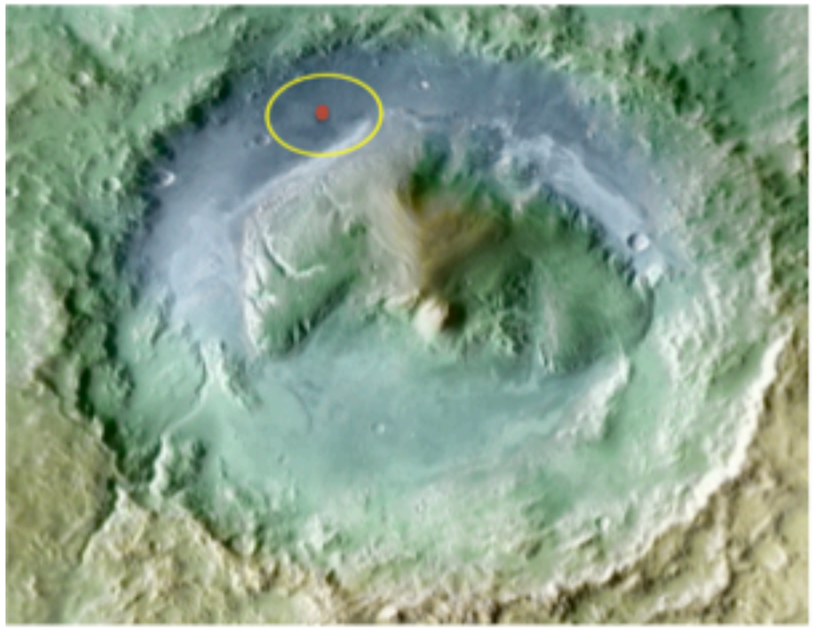
Take Home Message

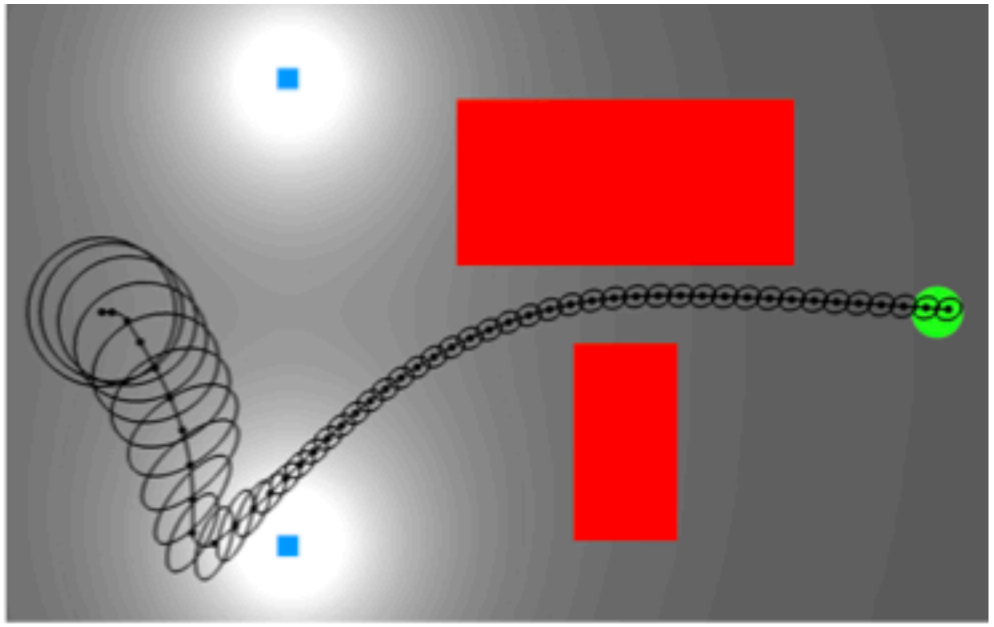
Emerging systems-control theory of PDFs

Three problems involving PDFs: prediction, filtering, control

One unifying framework: proximal recursion on the manifold of PDFs

Many applications:





Thank You