Reach Sets

Reach set for Integrator Dynamics

Integrator dynamics:
\[
\dot{x} = Ax + bu, \quad x \in \mathbb{R}^d, \quad u \in [-\mu, \mu]
\]
\[
A = \begin{bmatrix} 0 & e_1 & e_2 & \cdots & e_{d-1} \end{bmatrix}, \quad b = e_d
\]

The reach set with the set of initial conditions \( x_0 \)
\[
\mathcal{R}(x_0, t) = \exp(tA)x_0 + \int_0^t \exp(sA)b[-\mu, \mu]ds
\]

Closed-form Formula

Volume:
\[
\text{vol}(\mathcal{R}(x_0, t)) = (2\mu)^d t^{\frac{d(d+1)}{2}} \prod_{k=1}^{d-1} \frac{k!}{(2k + 1)!}
\]

Width in the direction \( \eta \in \mathbb{R}^{d-1} \)
\[
w_{\mathcal{R}}(x_0, t)(\eta) = 2\mu \int_0^t |\langle \eta, \xi(s) \rangle| \, ds
\]

where
\[
\xi(s) := \begin{pmatrix} s^{d-1} \frac{(d-1)!}{(d-2)!} \cdots s \end{pmatrix}^T
\]

Diameter = maximal width
\[
\text{diam}(\mathcal{R}(x_0, t)) = \max_{\eta \in \mathbb{S}^{d-1}} w_{\mathcal{R}}(x_0, t)(\eta)
\]

Ongoing work: Computing throughput for multi-agent dynamics
Details: arXiv: 1909.12498

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