Understanding the Geometry of Integrator Reach Sets for Robotics Applications

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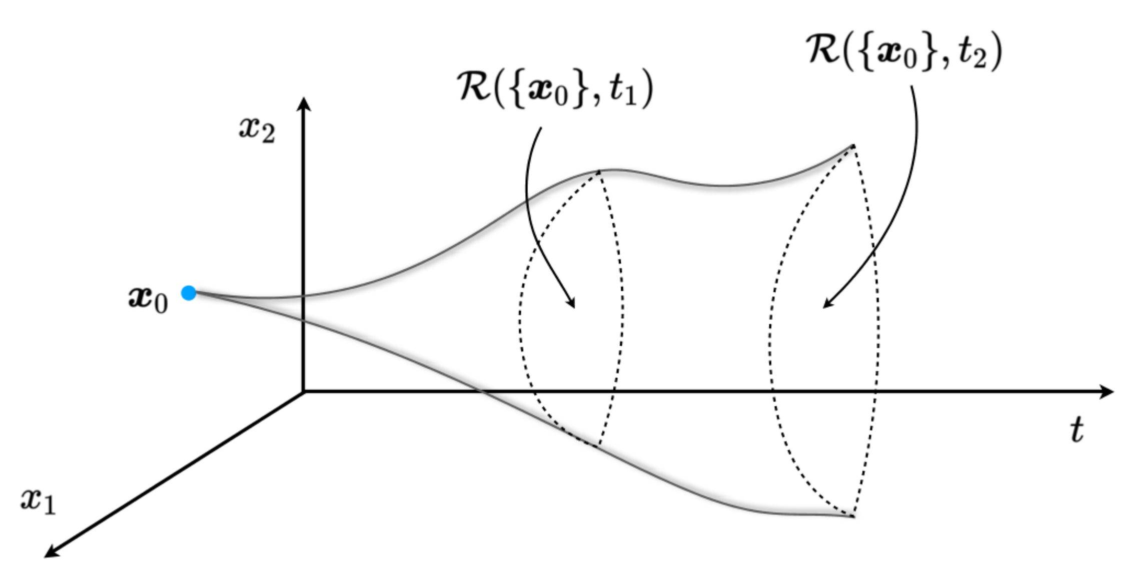
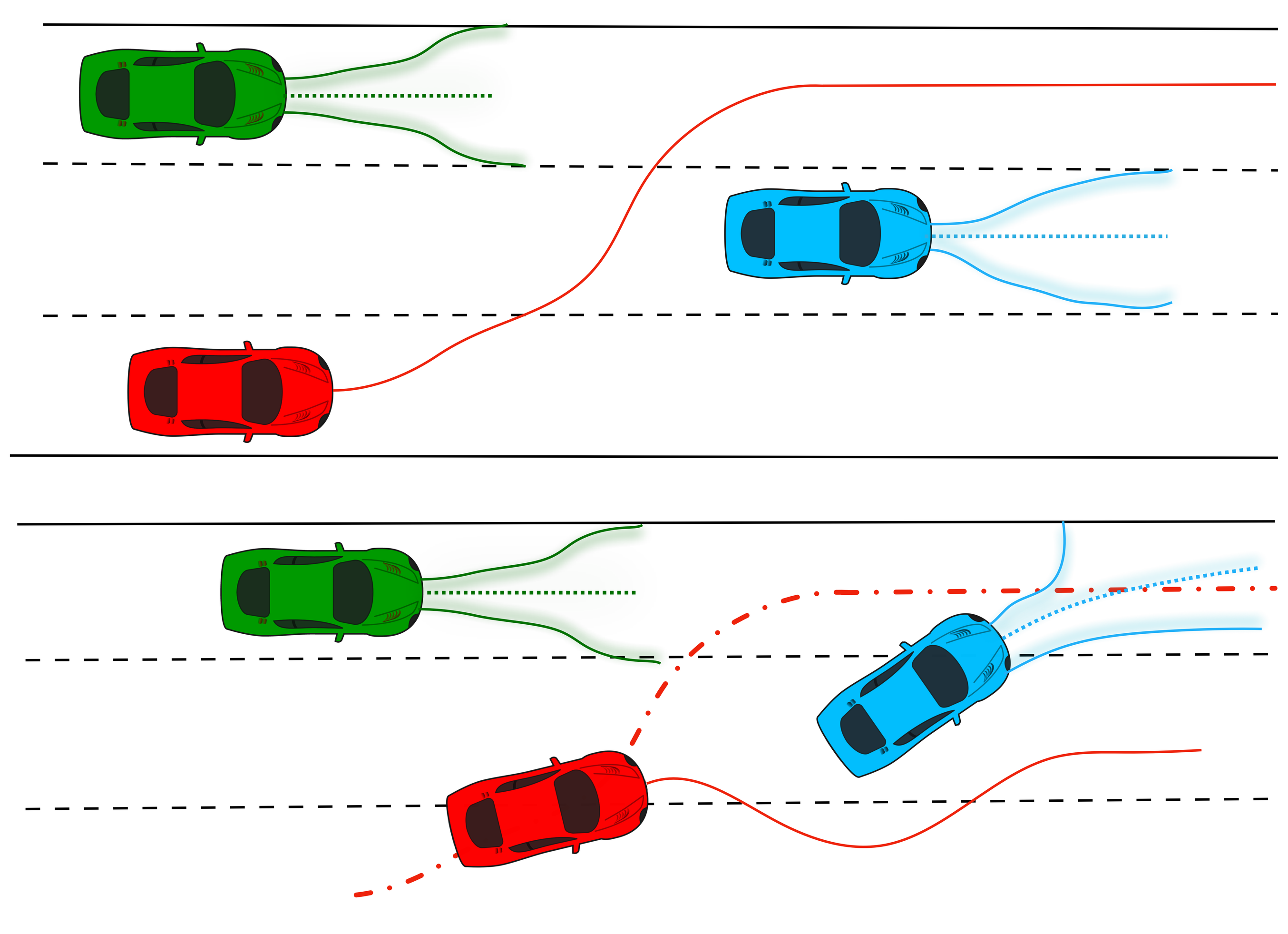
Reach Sets



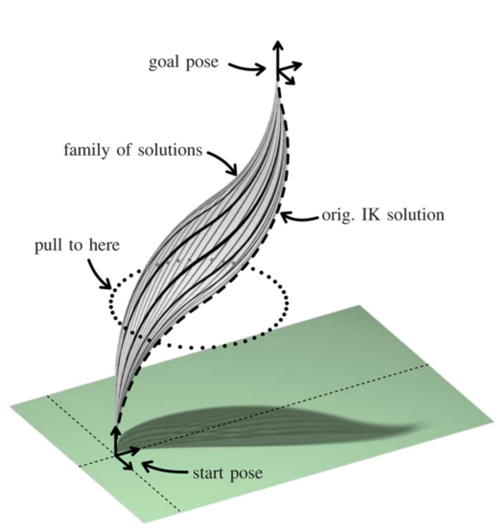
Where can the robot be at a future time subject to the dynamics and current knowledge of uncertainties

Safety-critical Applications

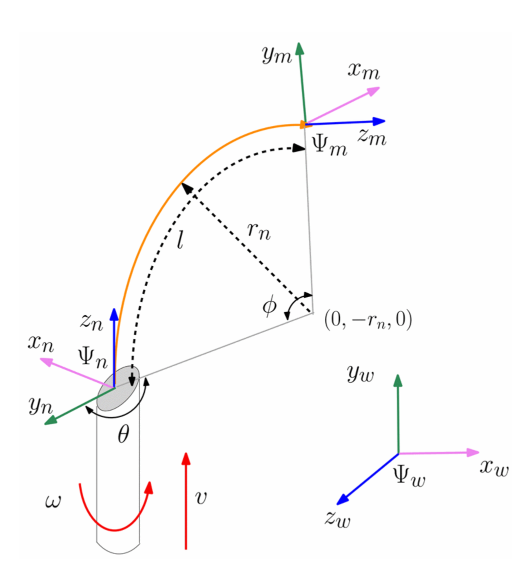
Collision avoidance



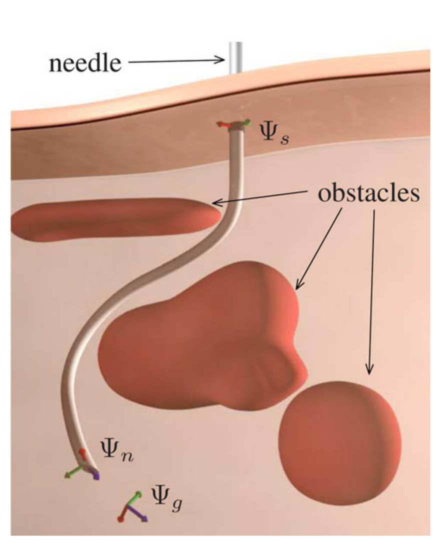
Needle steering



Credit: Duindam *et al.*, 2009



Credit: Patil and Alterovitz, 2010



Credit: Duindam *et al.*, 2009

Reach Set for Integrator Dynamics

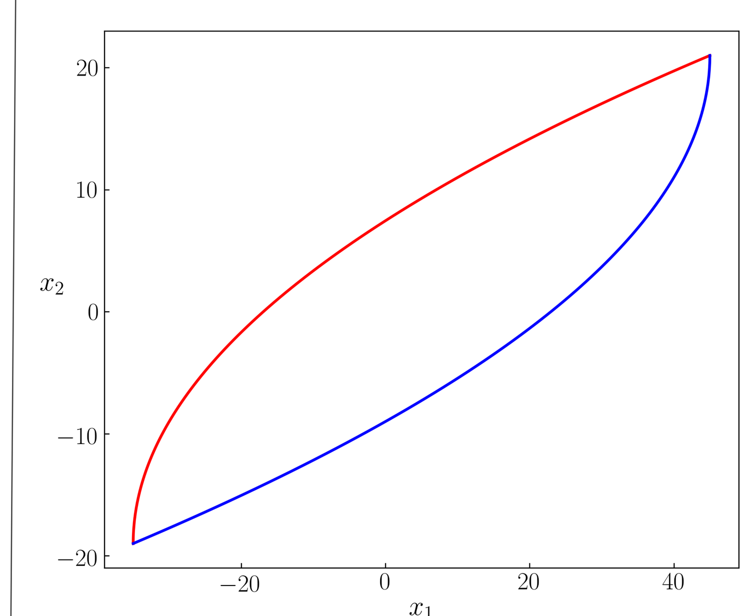
Integrator dynamics:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{x} \in \mathbb{R}^d, \quad u \in [-\mu, \mu]$$

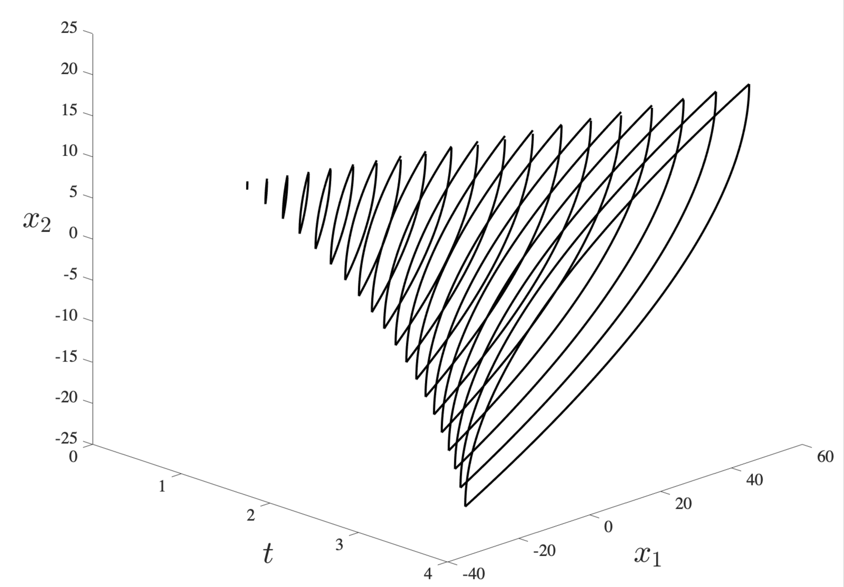
$$\mathbf{A} = [\mathbf{0} \quad \mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3 \quad \ldots \quad \mathbf{e}_{d-1}], \quad \mathbf{b} = \mathbf{e}_d$$

The reach set with the set of initial conditions $\mathcal{X}_0$

$$\mathcal{R}(\mathcal{X}_0, t) = \exp(t\mathbf{A})\mathcal{X}_o + \int_0^t \exp(s\mathbf{A})\mathbf{b}[-\mu, \mu]\mathrm{d}s$$



Reach set



Reachable tube

Size of the Integrator Reach Set

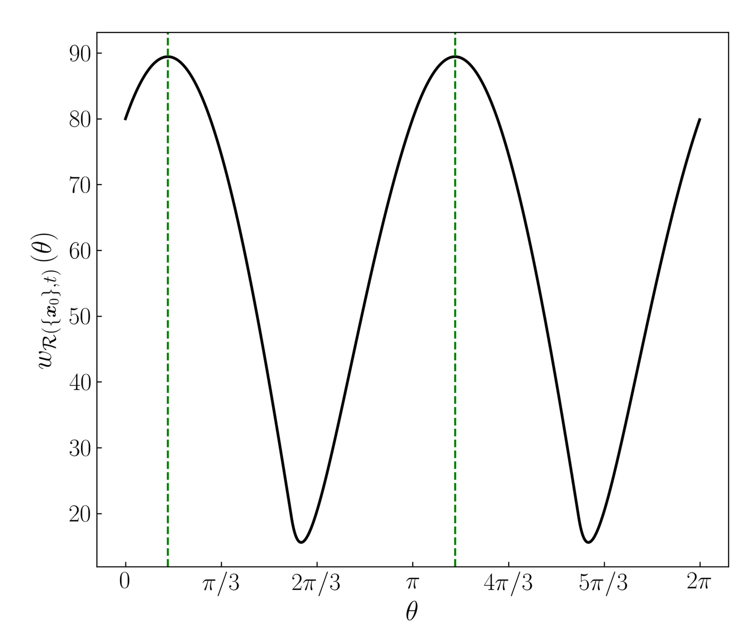
Volume $\mathrm{vol}\big(\mathcal{R}(\{\boldsymbol{x}_0\}, t)\big)$

Width in the direction $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$

$$w_{\mathcal{R}(\{\boldsymbol{x}_0, t\})}(\boldsymbol{\eta}) = 2\mu \int_0^t |\langle \boldsymbol{\eta}, \boldsymbol{\xi}(s)\rangle| \,\mathrm{d}s$$

where

$$\boldsymbol{\xi}(s) := \left(\frac{s^{d-1}}{(d-1)!} \quad \frac{s^{d-2}}{(d-2)!} \quad \cdots \quad s \quad 1\right)^\top$$



Diameter = maximal width

$$\mathrm{diam}\left(\mathcal{R}\left(\mathcal{X}_0, t\right)\right) = \max_{\boldsymbol{\eta} \in \mathbb{S}^{d-1}} w_{\mathcal{R}(\mathcal{X}_0, t)}(\boldsymbol{\eta})$$

Research Questions

How to compute the exact volume and diameter?

How do the volume and diameter scale with time? with dimension?

Closed-form Formula

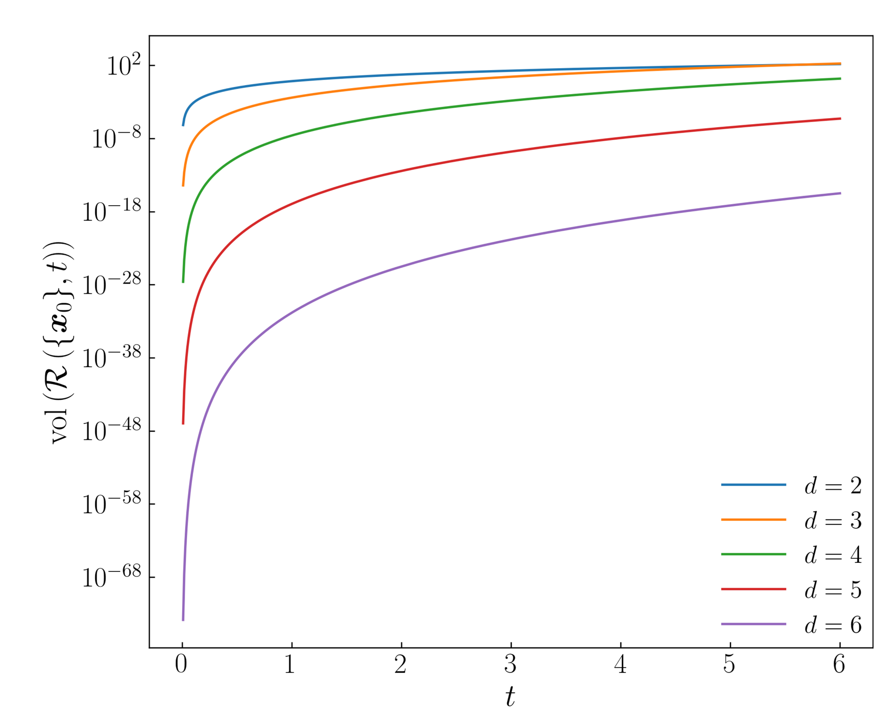
$$\mathrm{vol}\big(\mathcal{R}(\{\boldsymbol{x}_0\}, t)\big) = (2\mu)^d t^{\frac{d(d+1)}{2}} \prod_{k=1}^{d-1} \frac{k!}{(2k+1)!}$$

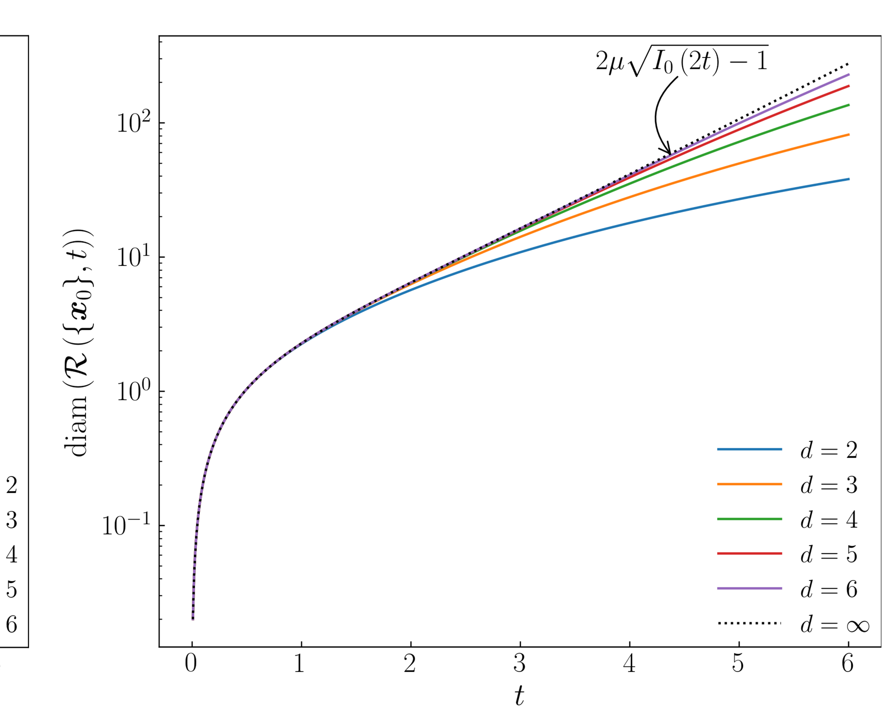
$$\mathrm{diam}\big(\mathcal{R}(\{\boldsymbol{x}_0\}, t)\big) = 2\mu \sqrt{\sum_{j=1}^{d} \left(\frac{t^j}{j!}\right)^2}$$

Key concepts:

The reach set $\mathcal{R}(\{\boldsymbol{x}_0\}, t)$ is a **zonoid** (limit of a sequence of **zonotopes**)

A **zonotope** is the Minkowski sum of line intervals





Ongoing work: Computing throughput for multi-agent dynamics

Details: arXiv: 1909.12498

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