Anytime Ellipsoidal Over-approximation of Forward Reach Sets of Uncertain Linear Systems

Shadi Haddad, Abhishek Halder

University of California, Santa Cruz

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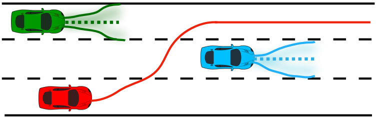
V&V computation for safety-critical CPS

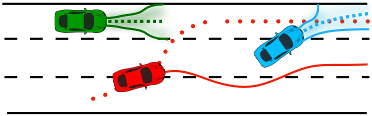
- compute provably tight outer-approximation of the forward reach sets for closed-loop dynamics

- "outer" $\rightsquigarrow$ "safe"

- "tight" $\rightsquigarrow$ "least conservative" (e.g., min volume)

- natural ways to account set-valued uncertainties





Computationally demanding in general ...

- **Nonparametric:** level set toolbox [Mitchell]

- **Parametric:** ellipsoidal toolbox [Kurzhanskiy-Varaiya], CORA [Althoff], many others

- **Semiparametric:** data-driven reachability, growing literature in last 3-5 years

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Safety-critical CPS platforms typically have scarce computational resources

Natural idea: anytime over-approximation

- provable over-approximation
- monotonically adapt tightness w.r.t. computational time available

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This talk: Anytime ellipsoidal over-approximation for linear systems + set-valued uncertainties

Ellipsoids

$(\boldsymbol{q}, \boldsymbol{Q}) \in \mathbb{R}^d \times \mathbb{S}^d_{++}$ **parameterization:**

$$\mathcal{E}(\boldsymbol{q}, \boldsymbol{Q}) := \left\{ \boldsymbol{y} \in \mathbb{R}^d \mid (\boldsymbol{y} - \boldsymbol{q})^\top \boldsymbol{Q}^{-1} (\boldsymbol{y} - \boldsymbol{q}) \leq 1 \right\}$$

$(\boldsymbol{A}_0, \boldsymbol{b}_0, c_0) \in \mathbb{S}^d_{++} \times \mathbb{R}^d \times \mathbb{R}$ **parameterization:**

$$\mathcal{E}\left(\boldsymbol{A}_0, \boldsymbol{b}_0, c_0\right) := \left\{ \boldsymbol{y} \in \mathbb{R}^d \mid \boldsymbol{y}^\top \boldsymbol{A}_0 \boldsymbol{y} + 2\boldsymbol{y}^\top \boldsymbol{b}_0 + c_0 \leq 1 \right\}$$

Why ellipsoids

- **fixed parameterization complexity:**
  need $d(d+3)/2$ reals in d dimensions

- **natural for modeling:**
  weighted norm-bounded uncertainties $\sim$ time-varying ellipsoids

- **mathematically nice:**
  minimum volume outer ellipsoid (MVOE) a.k.a. Löwner-John ellipsoid $\mathcal{E}_{\mathrm{LJ}}$ of any compact set in unique

Models

Linear system:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u} + \boldsymbol{G}(t)\boldsymbol{w}$$

Uncertainties:

$$\boldsymbol{x}(0) \in \mathcal{X}_0 := \mathcal{E}\left(\boldsymbol{x}_0, \boldsymbol{X}_0\right)$$

$$\boldsymbol{u} \in \mathcal{U}(t) := \mathcal{E}\left(\boldsymbol{u}_c(t), \boldsymbol{U}(t)\right)$$

$$\boldsymbol{w} \in \mathcal{W}(t) := \mathcal{E}\left(\boldsymbol{w}_c(t), \boldsymbol{W}(t)\right)$$

Forward reach set:

$$\mathcal{R}\left(\mathcal{X}_0, t\right) := \{\boldsymbol{x}(t) \in \mathbb{R}^n \mid \dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u} + \boldsymbol{G}(t)\boldsymbol{w}$$
$$\boldsymbol{x}(0) \in \boldsymbol{X}_0, \boldsymbol{u} \in \mathcal{U}(t), \boldsymbol{w} \in \mathcal{W}(t)\}$$

Ellipsoidal over-approximation of $\mathcal{R}(\mathcal{X}_0, t)$

- Due to Kurzhanski and Varaiya

- Construct a family $\{\mathcal{E}(\boldsymbol{x}_c(t), \boldsymbol{X}_i(t))\}_{i=1}^{N}$ parameterized by unit vectors $\boldsymbol{\ell}_{10}, \ldots, \boldsymbol{\ell}_{N0} \in \mathbb{R}^n$ such that

  $$\mathcal{R}(\mathcal{X}_0, t) \subseteq \widehat{\mathcal{R}}_N(\mathcal{X}_0, t) := \bigcap_{i=1}^{N} \mathcal{E}(\boldsymbol{x}_c(t), \boldsymbol{X}_i(t))$$

  for any finite $N = 1, 2, \ldots$

- Also, $\bigcap_{i=1}^{\infty} \mathcal{E}(\boldsymbol{x}_c(t), \boldsymbol{X}_i(t)) = \mathcal{R}(\mathcal{X}_0, t)$

Constructing $\widehat{\mathcal{R}}_N$

- Let $\boldsymbol{\ell}_i(t) := \exp\left(-(\boldsymbol{A}(t))^\top t\right) \boldsymbol{\ell}_{i0}$

  and $$\pi_i(t) := \left(\frac{\boldsymbol{\ell}_i^\top(t)\boldsymbol{B}(t)\boldsymbol{U}(t)\boldsymbol{B}^\top(t)\boldsymbol{\ell}_i(t)}{\boldsymbol{\ell}_i^\top(t)\boldsymbol{X}_i(t)\boldsymbol{\ell}_i(t)} \right)^{1/2}$$

- Find orthogonal $\boldsymbol{S}_i(t)$ such that

$$\boldsymbol{S}_i(t)\frac{\boldsymbol{X}_i^{1/2}(t)\boldsymbol{\ell}_i(t)}{\left\|\boldsymbol{X}_i^{1/2}(t)\boldsymbol{\ell}_i(t)\right\|_2} = \frac{\boldsymbol{G}(t)\boldsymbol{W}(t)\boldsymbol{G}^\top(t)\boldsymbol{\ell}_i(t)}{\left\|\boldsymbol{G}(t)\boldsymbol{W}(t)\boldsymbol{G}^\top(t)\boldsymbol{\ell}_i(t)\right\|_2}$$

Construct $\mathcal{E}\left(x_c(t), X_i(t)\right)$ by solving

Center vector initial value problem:

$$\dot{\boldsymbol{x}}_c(t) = \boldsymbol{A}(t)\boldsymbol{x}_c(t) + \boldsymbol{B}(t)\boldsymbol{u}_c(t) + \boldsymbol{G}(t)\boldsymbol{w}_c(t),\ \boldsymbol{x}_c(0) = \boldsymbol{x}_0$$

Shape matrix initial value problem:

$$\begin{aligned}\dot{\boldsymbol{X}}_i(t) &= \boldsymbol{A}(t)\boldsymbol{X}_i(t) + \boldsymbol{X}_i(t)(\boldsymbol{A}(t))^\top + \pi_i(t)\boldsymbol{X}_i(t) \\ &\quad + \frac{1}{\pi_i(t)}\boldsymbol{B}(t)\boldsymbol{U}(t)\boldsymbol{B}^\top(t) - \boldsymbol{X}_i^{1/2}(t)\boldsymbol{S}_i(t)\boldsymbol{G}(t)\boldsymbol{W}(t)\boldsymbol{G}^\top(t) \\ &\quad - \boldsymbol{G}(t)\boldsymbol{W}(t)\boldsymbol{G}^\top(t)\boldsymbol{S}_i^\top(t)\boldsymbol{X}_i^{1/2}(t), \quad \boldsymbol{X}_i(0) = \boldsymbol{X}_0\end{aligned}$$

Wanted: MVOE $\mathcal{E}\left(\boldsymbol{x}_c(t), \boldsymbol{X}(t)\right) \supseteq \widehat{\mathcal{R}}_N$

$$\begin{aligned} \underset{\boldsymbol{X}(t) \succ \boldsymbol{0}}{\arg\min} \quad & \operatorname{vol}\left(\mathcal{E}\left(\boldsymbol{x}_c(t), \boldsymbol{X}(t)\right)\right) \\ \text{s.t.} \quad & \textstyle\bigcap_{i=1}^{N} \mathcal{E}\left(\boldsymbol{x}_c(t), \boldsymbol{X}_i(t)\right) \subseteq \mathcal{E}\left(\boldsymbol{x}_c(t), \boldsymbol{X}(t)\right) \end{aligned}$$

- convex semi-infinite program

- verifying the constraint for $N+1$ given ellipsoids is NP complete

Relaxation based on S procedure

$$\begin{aligned}
&\underset{\widetilde{A},\widetilde{b},\tau_1,\ldots,\tau_N}{\text{minimize}} \quad \log\det \widetilde{A}^{-1} \\
&\text{s.t.} \quad \widetilde{A} \succ \mathbf{0}, \quad \tau_1,\ldots,\tau_N \geq 0, \\
&\begin{bmatrix} \widetilde{A} & \widetilde{b} & \mathbf{0} \\ \widetilde{b}^\top & -1 & \widetilde{b}^\top \\ \mathbf{0} & \widetilde{b} & -\widetilde{A} \end{bmatrix} - \textstyle\sum_{i=1}^N \tau_i \begin{bmatrix} A_i & b_i & 0 \\ b_i^\top & c_i & 0 \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix} \preceq \mathbf{0}
\end{aligned}$$

return $\mathcal{E}\left(-\widetilde{A}_{\text{opt}}^{-1}\widetilde{b}_{\text{opt}}, \widetilde{A}_{\text{opt}}^{-1}\right)$
in (q, Q) parameterization

Speeding up computation

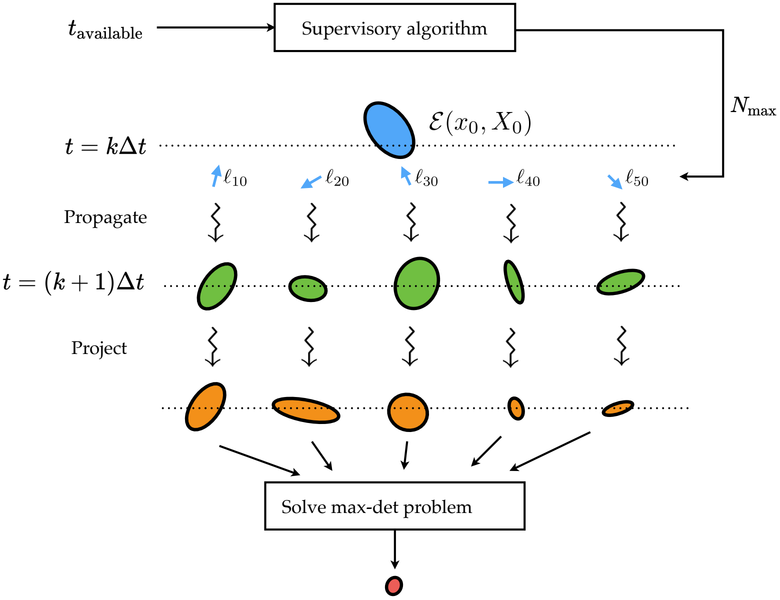
- Propagation of ellipsoids $\rightsquigarrow$ solve $N+1$ initial value problems in parallel

- Projection:

$$\text{proj}\left(\mathcal{E}_{\text{LJ}}\left(\bigcap_{i=1}^{N}\mathcal{E}\left(\boldsymbol{x_c}(t),\boldsymbol{X}_i(t)\right)\right)\right)=\mathcal{E}_{\text{LJ}}\left(\text{proj}\left(\bigcap_{i=1}^{N}\mathcal{E}\left(\boldsymbol{x_c}(t),\boldsymbol{X}_i(t)\right)\right)\right)$$

$$\subseteq \mathcal{E}_{\text{LJ}}\left(\bigcap_{i=1}^{N}\text{proj}\left(\mathcal{E}\left(\boldsymbol{x}_c(t),\boldsymbol{X}_i(t)\right)\right)\right)\subseteq \begin{array}{c}\text{minimizer of maxdet problem}\\ \text{w. input proj}(\cdot)\text{ of }\mathcal{E}\left(\boldsymbol{x}_c(t),\boldsymbol{X}_i(t)\right)\end{array}$$

Anytime computation



Anytime computation

- At $t = k\Delta t$, we have $t_{\text{available}} < \Delta t$ time available to compute $\mathcal{R}\left(\mathcal{X}_0, t = (k+1)\Delta t\right)$

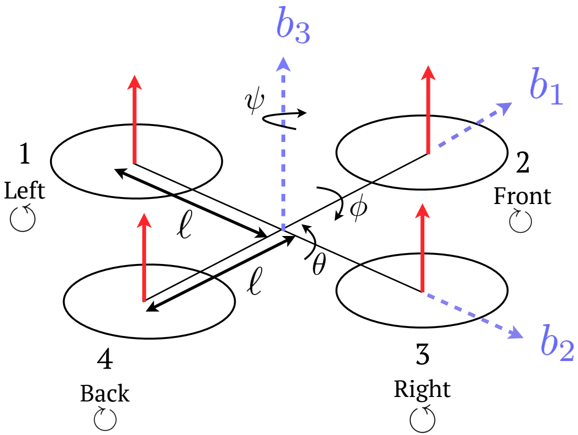
- $t_{\text{available}}$ depends on processor availability

- $t_{\text{propagation}} + t_{\text{opt}} = f(N)$, estimate $\widehat{f}$ from data and find maximal real root of $t_{\text{available}} = \widehat{f}(\widehat{N})$. Then $N_{\max} = \lfloor \widehat{N} \rfloor$

- May also learn $N_{\max}$ online

Numerical case study: controlled quadrotor

- $n = 12$ states, $m = 4$ inputs, $p = 3$ unmeasured disturbances



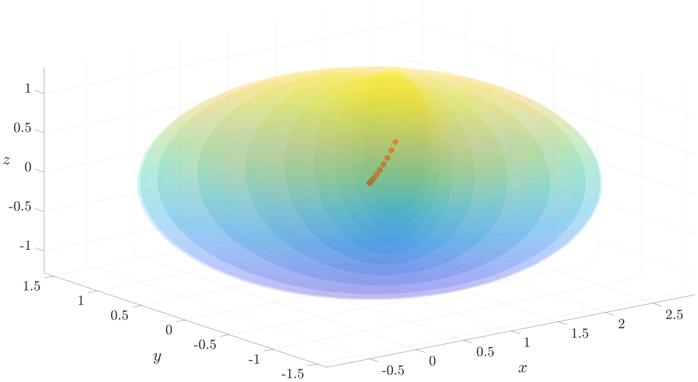
Numerical case study: controlled quadrotor

- closed-loop LTV dynamics with finite horizon LQ tracker + estimation error

- $\dot{\boldsymbol{x}} = \boldsymbol{A}_{\text{cl}}(t)\boldsymbol{x} + \boldsymbol{B}_{\text{cl}}\boldsymbol{\eta} + \boldsymbol{G}\boldsymbol{w}$,
  $\boldsymbol{x}(0) \in \mathcal{E}\left(\boldsymbol{x}_0, \boldsymbol{X}_0\right)$,
  $\boldsymbol{\eta} \in \mathcal{E}\left(\boldsymbol{v}(t), \boldsymbol{P}(t)\boldsymbol{E}(t)\boldsymbol{P}^\top(t)\right)$,
  $\boldsymbol{w} \in \mathcal{E}\left(\boldsymbol{w}_c(t), \boldsymbol{W}(t)\right)$

Numerical case study: controlled quadrotor

- ellipsoidal over-approximation in (x, y, z) position coordinates: 10 snapshots in $t \in [0, 1]$ with $N_{\max} = 10$



Summary of findings

- Anytime implementation for ellipsoidal over-approximation
- Computational time dominated by ellipsoidal propagation
- Possible directions: anytime algorithms for other parametric/nonparametric/semiparametric algorithms, online learning for supervisor

Thank You

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