Aero 320: Numerical Methods

Homework 4

Name: ....

Due: October 21, 2013

NOTE: All problems, unless explicitly asked to write a code, are to be done by hand (with the help of a calculator) but **you need to show all the steps**. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Submit your HW by Monday midnight at Room 201, Reed McDonald Building. Late submissions or failure to submit in the required format will receive no credit.

Problem 1

LU decomposition $(5+3+2+5+2+6+2=25$ **points)**

Let $A = \begin{pmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix}.$

(a) By hand, perform LU decomposition for matrix A. Show all the calculations in *exact arithmetic* (i.e. use fractions throughout).

(b) Use your answer in part (a) to compute $\det(A)$.

(c) From your answer to part (b), what can you say about the solution for the system of linear equations given by $Ax = b$?

(d) Use the LU decomposition from part (a), to solve for vector x such that $Ax = b$.

(e) A matrix is invertible if it has non-zero determinant. From part (b), does A^{-1} exist?

(f) If exists, then A^{-1} can be computed by solving the matrix equation $AX = I$ for square matrix X, where I is the identity matrix of size same as A. Use the LU decomposition from part (a), to compute A^{-1}.

(g) From your answer in part (f), find $x = A^{-1}b$. Compare your result with that found in part (d). Why (d) is a better algorithm to solve $Ax = b$ than directly computing $x = A^{-1}b$, even though $\det(A) \neq 0$?

Solution

(a) LU decomposition:

$$\begin{pmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{pmatrix} \xrightarrow{\text{Row 2=Row 2}-\frac{1}{2}\text{Row 1}} \begin{pmatrix} 2 & -2 & 4 \\ \textcircled{\tfrac{1}{2}} & -2 & -1 \\ 3 & 7 & 5 \end{pmatrix} \xrightarrow{\text{Row 3=Row 3}-\frac{3}{2}\text{Row 1}} \begin{pmatrix} 2 & -2 & 4 \\ \textcircled{\tfrac{1}{2}} & -2 & -1 \\ \textcircled{\tfrac{3}{2}} & 10 & -1 \end{pmatrix}$$

$$\xrightarrow{\text{Row 3=Row 3}-(-5)\text{Row 2}} \begin{pmatrix} 2 & -2 & 4 \\ \textcircled{\tfrac{1}{2}} & -2 & -1 \\ \textcircled{\tfrac{3}{2}} & \textcircled{-5} & -6 \end{pmatrix}.$$

Hence, $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{pmatrix}.$$

(b) $\det(A) = \det(LU) = \det(L)\det(U) = \det(U) = 2 \times (-2) \times (-6) = 24.$

(c) Since $\det(A) \neq 0$, the system of linear equations $Ax = b$ has unique solution.

(d) First, we solve for $Ly = b$, using *forward substitution*. This results $y = \begin{pmatrix} 0 \\ -5 \\ -18 \end{pmatrix}$. Next, we solve $Ux = y$, using *backward substitution*. This results $x = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$.

(e) Since $\det(A) \neq 0$ (from part (b)), the matrix A^{-1} exists.

(f) Solving the 3×3 matrix equation $AX = I$ is same as solving three matrix-vector equations

$$AX_1 = e_1, \quad AX_2 = e_2, \quad AX_3 = e_3,$$

where X_i is the i^{th} column of X, and e_i is the i^{th} column of I. For example, $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, etc. We can solve each matrix-vector equation $AX_i = e_i$, using LU decomposition of matrix A. Proceeding similar to part (d), we get $X_1 = \begin{pmatrix} -\frac{11}{12} \\ -\frac{1}{12} \\ \frac{2}{3} \end{pmatrix}$, $X_2 = \begin{pmatrix} \frac{19}{12} \\ -\frac{1}{12} \\ -\frac{5}{6} \end{pmatrix}$, $X_3 = \begin{pmatrix} \frac{5}{12} \\ \frac{1}{12} \\ -\frac{1}{6} \end{pmatrix}$. Now, we can stack these columns to write

$$A^{-1} = \begin{pmatrix} -\frac{11}{12} & \frac{19}{12} & \frac{5}{12} \\ -\frac{1}{12} & -\frac{1}{12} & \frac{1}{12} \\ \frac{2}{3} & -\frac{5}{6} & -\frac{1}{6} \end{pmatrix}.$$

(g) Using the answer from part (f), we perform the matrix-vector multiplication:

$$A^{-1}b = \begin{pmatrix} -\frac{11}{12} & \frac{19}{12} & \frac{5}{12} \\ -\frac{1}{12} & -\frac{1}{12} & \frac{1}{12} \\ \frac{2}{3} & -\frac{5}{6} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{-95+35}{12} \\ \frac{5+7}{12} \\ \frac{25-7}{6} \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix},$$

which is same as the answer we got in part (d). However, part (d) is a better algorithm since there we did single LU decomposition; but to do $A^{-1}b$, we had to compute A^{-1} in part (f), which itself required three LU decompositions. Thus, computing $x = A^{-1}b$ is three times (for $n \times n$ matrix, n times) costlier than solving for x via direct LU decomposition.

Problem 2

Vector and matrix norms **(3+3+4+4+6 = 20 points)**

(a) By hand, compute the 1-norm, 2-norm and ∞-norm of the vector $x = \{-\sqrt{3} \quad -6 \quad 4 \quad 2\}^\top$.

(b) For any $n \times 1$ vector x, the following holds:

$$\| x \|_\infty \quad \leq \quad \| x \|_2 \quad \leq \quad \| x \|_1 \quad \leq \quad \sqrt{n} \| x \|_2 \quad \leq \quad n \| x \|_\infty .$$

Verify this relation for the vector in part (a).

(c) By hand, compute the 1-norm, 2-norm, ∞-norm, and Frobenius norm of the matrix

$$M = \begin{pmatrix} 3 & 5 & 7 \\ 2 & -6 & 4 \\ -1 & 2 & 8 \end{pmatrix}.$$

(d) Consider any $n \times n$ orthogonal matrix Q. Compute $\| Q \|_2$, $\| Q \|_F$.

(e) Give examples of matrix A such that (i) $\| A \|_1 < \| A \|_\infty$, (ii) $\| A \|_1 = \| A \|_\infty$, and (iii) $\| A \|_1 > \| A \|_\infty$.

Solution

(a)

$$\| x \|_1 = \sum_{i=1}^{4} |x_i| = 12 + \sqrt{3}, \qquad \| x \|_2 = \sqrt{\sum_{i=1}^{4} x_i^2} = \sqrt{59}, \qquad \| x \|_\infty = \max_{i=1,\ldots,4} |x_i| = 6.$$

(b) The inequalities hold for our case since

$$6 \quad \leq \quad \sqrt{59} \approx 7.6811 \quad \leq \quad 12 + \sqrt{3} \approx 13.7321 \quad \leq \quad \sqrt{4} \times \sqrt{59} = 15.3623 \quad \leq \quad 4 \times 6 = 24.$$

(c)

$$\| M \|_1 = \max_{j=1,\ldots,3} \sum_{i=1}^{3} |m_{ij}| = \max_{j=1,\ldots,3} \{6, 13, 19\} = 19, \qquad \| M \|_\infty = \max_{i=1,\ldots,3} \sum_{j=1}^{3} |m_{ij}| = \max_{j=1,\ldots,3} \{15, 12, 11\} = 15,$$

$$\| M \|_F = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3} m_{ij}^2} = 14.4222 \text{ (can also be computed as } \sqrt{\operatorname{tr}(MM^\top)}), \qquad \| M \|_2 = \sqrt{\lambda_{\max}(MM^\top)} = 11.9174.$$

To obtain the 2-norm, we observe that the characteristic equation for matrix $MM^\top = \begin{pmatrix} 83 & 4 & 63 \\ 4 & 56 & 18 \\ 63 & 18 & 69 \end{pmatrix}$ is a cubic polynomial, which can be solved (by hand/by calculator/by root solving code in earlier assignments) to get $\lambda_{\max} = 11.9174$. You may verify your answers with MATLAB commands `norm(M,1)`, `norm(M,2)`, `norm(M,inf)`, `norm(M,'fro')`.

(d) For any $n \times n$ orthogonal matrix Q, we know $QQ^\top = I$. Hence, $\| Q \|_2 = \sqrt{\lambda_{\max}(QQ^\top)} = \sqrt{\lambda_{\max}(I)} = 1$. Similarly, $\| Q \|_F = \sqrt{\operatorname{tr}(QQ^\top)} = \sqrt{\operatorname{tr}(I)} = \sqrt{n}$.

(e) (i) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then $\| A \|_1 = 6 < \| A \|_\infty = 7$. (ii) Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Then $\| A \|_1 = \| A \|_\infty = 6$. This will happen for any symmetric matrix. (iii) Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. Then $\| A \|_1 = 7 > \| A \|_\infty = 6$.

Problem 3

Condition number and ill-conditioned problems $\qquad$ **($11 + 5 + (5 + 3 + 1) = 25$ points)**

The *condition number* $\kappa_*(A)$ of a matrix A, is defined as $\kappa_*(A) = \| A \|_* \| A^{-1} \|_*$, where $*$ is any matrix norm. For example, if we use 2-norm of matrix, then we get $\kappa_2(A) = \| A \|_2 \| A^{-1} \|_2$, etc. A system of linear equations of the form $Ax = b$, is said to be *ill-conditioned* if small changes in A or b produce large changes in the solution x.

(a) Consider solving $Ax = b$, where $A = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix}$. You may verify that $\det(A) = 1$, and hence there is unique solution. Using your favorite method (Gauss elimination or LU) solve for x when (i) $b = \{23 \quad 32 \quad\quad 33 \quad 31\}^\top$, (ii) $b = \{22.9 \quad 32.1 \quad 32.9 \quad 31.1\}^\top$, (iii) $b = \{22.99 \quad 32.01 \quad 32.99 \quad 31.01\}^\top$. Try to compute your answers as accurately as you can. What do you think is happening?

(b) Compute $\kappa_\infty(A)$ for the matrix in part (a).

(c) A Hilbert matrix H of size $n \times n$ has entries: $H_{ij} = \dfrac{1}{i+j-1}$. Write a C++ code to compute $\kappa_2(H)$ for $n = 2, 4, 8, 16, 32$. Submit a hard copy of your code, and a plot of $\kappa_2(H)$ versus n. What is your conclusion from this plot?

Solution

(a) Use any method (Gauss elimination or LU decomposition) in exact arithmetic, to get the solution as: (i) $x = \{1 \quad 1 \quad 1 \quad 1\}^\top$, (ii) $x = \{-7.2 \quad 6 \quad 2.9 \quad 0.1\}^\top$, (iii) $x = \{0.18 \quad 1.5 \quad 1.19 \quad 0.89\}^\top$. We see that small changes in vector b cause large changes in the solution x, meaning the system of equations given by $Ax = b$, is ill-conditioned.

(b) Using LU decomposition for matrix A, as in Problem 1(f), we can compute $A^{-1} = \begin{pmatrix} 68 & -41 & -17 & 10 \\ -41 & 25 & 10 & -6 \\ -17 & 10 & 5 & -3 \\ 10 & -6 & -3 & 2 \end{pmatrix}$.

Then, we have $\| A \|_\infty = 33$, and $\| A^{-1} \|_\infty = 136$. Hence, $\kappa_\infty(A) = \| A \|_\infty \| A^{-1} \|_\infty = 33 \times 136 = 4488$. Notice that this number $\kappa_\infty(A)$ is much larger than 1, implying the system of equations of the form $Ax = b$ will be ill-conditioned, a fact that we already verified in part (a).

(c) See code attached. Run the C++ code (`Homework4Problem3c.cpp` file) followed by the MATLAB code

(`HW4Problem3c.m` file). The conclusion from the plot is this: larger the dimension, more ill-conditioned the Hilbert matrix is.

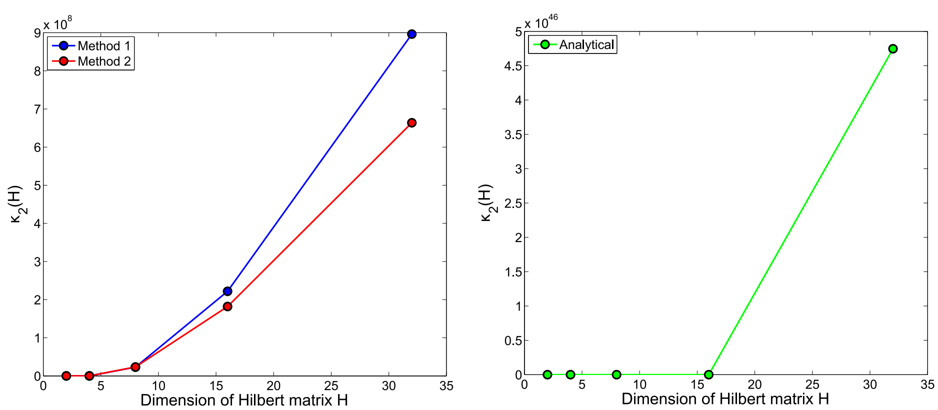

Figure 1: Condition number $\kappa_2(H)$ of Hilbert matrix H computed in different ways: numerical on left, and exact arithmetic on right. See the MATLAB code for details on the methods. All plots confirm the trend that the Hilbert matrix becomes more ill-conditioned as the size of the matrix increases.

It can be observed that there are significant numerical errors (orders of magnitude mismatch) for this approach of Hilbert matrix construction in C++, followed by numerical inversion or eigenvalue solving in MATLAB. The question then becomes: how can we write a C++ code that will be as close to the analytical answer (`hilb` and `invhilb` implementations in MATLAB), as possible. The file `HilbCond2.cpp` achieves this by implementing power iteration for maximum eigenvalue (similar to Lab 12). This C++ code outputs the data file `HilbertCond2.dat`, that lists the condition number $\kappa_2(H)$ for different n. Notice how close this gets to MATLAB's prediction. Since the real trick is in computing the 2-norm, below we compare the norms between MATLAB's analytical implementation with answers obtained from `HilbCond2.cpp`.

n	$\parallel H \parallel_2$ from `HilbCond2.cpp`	`norm(hilb(n),2)` from **MATLAB**
2	1.26759	1.2676
4	1.50019	1.5002
8	1.69569	1.6959
16	1.85882	1.8600
32	1.99459	1.9984

n	$\parallel H^{-1} \parallel_2$ from `HilbCond2.cpp`	`norm(invhilb(n),2)` from **MATLAB**
2	15.2105	15.2111
4	1.0341×10^4	1.0341×10^4
8	8.99653×10^9	8.9965×10^9
16	1.08726×10^{22}	1.0873×10^{22}
32	1.37198×10^{46}	2.3752×10^{46}

Problem 4

Jacobi and Gauss-Seidel iteration $(10 + 4 + 4 + 4 + 8 = 30$ **points)**

In this exercise, you will see that for some problems, Jacobi method may converge with any initial guess, but the Gauss-Seidel method may fail.

(a) First, write C++ codes to iteratively solve the system of linear equations $Ax = b$ using *Jacobi method* and *Gauss-Seidel method*. If the k^{th} iterate is the vector x_k, then the convergence condition is that the *2-norm relative error* becomes less than the tolerance $\epsilon = 10^{-4}$, that is:

$$\frac{\parallel Ax_k - b \parallel_2}{\parallel b \parallel_2} < \epsilon.$$

Define the maximum number of iterations to be 500 to stop the code in case it diverges. Submit the hard copies for your codes.

(b) Test your codes in part (a), for $A = \begin{pmatrix} 3 & -5 & 2 \\ 5 & 4 & 3 \\ 2 & 5 & 3 \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Start with initial guess $x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Report, if possible, the number of iterations needed for each method to converge.

(c) How does your answer to part (b) change if we modify the convergence criterion to be

$\dfrac{\| Ax_k - b \|_\infty}{\| b \|_\infty} < \epsilon$? Explain the change, if any, you observe, compared to part (b).

(d) Repeat part (c) with the convergence criterion $\dfrac{\| Ax_k - b \|_1}{\| b \|_1} < \epsilon$.

(e) For part (b), (c), (d) above, plot the corresponding relative error versus iteration number k on the same figure.

Solution

See code attached. For the attached code, the convergence results are as follows:

Convergence criteria	**Jacobi method**	**Gauss-Seidel method**
1-norm	500 iterations	Diverges
2-norm	500 iterations	Diverges
∞-norm	85 iterations	Diverges

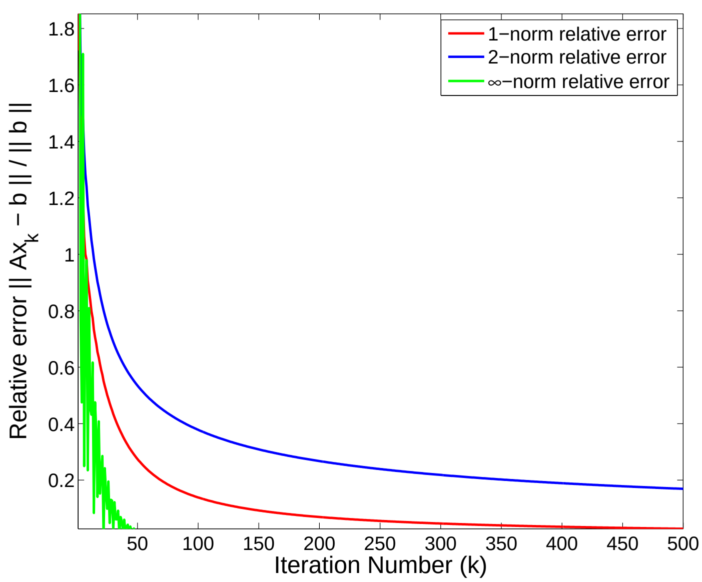

Figure 2: Convergence of Jacobi iteration in Problem 4 with different norm based relative error criteria.

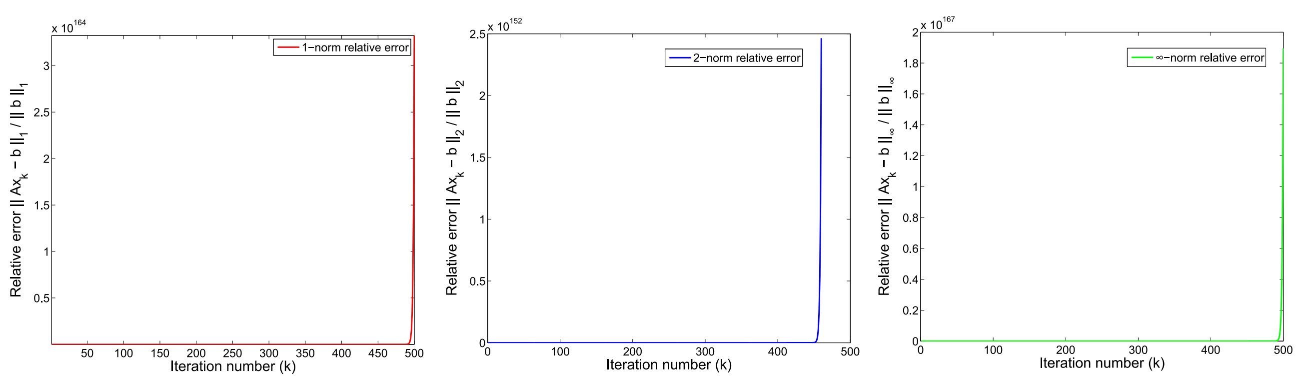

Figure 3: Convergence of Gauss-Seidel iteration in Problem 4 with different norm based relative error criteria.