Aero 320: Numerical Methods

Homework 5

Name: ....

Due: November 4, 2013

NOTE: All problems, unless explicitly asked to write a code, are to be done by hand (with the help of a calculator) but **you need to show all the steps**. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Submit your HW by Monday midnight at Room 201, Reed McDonald Building. Late submissions or failure to submit in the required format will receive no credit.

Problem 1

Interpolation using monomial basis and Lagrange polynomials
$(5 + 10 + 10 + 10 + 2 + 3 = 40$ **points)**

Suppose we have observed 4 experimental data points: $(x_0, y_0), \quad (x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3)$.

(a) If we want to interpolate these data using standard monomial basis, then we need to compute the coefficient vector $c = \{c_0 \quad c_1 \quad c_2 \quad c_3\}^\top$ such that the interpolating polynomial is $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. Find the matrix-vector equation from which vector c can be solved.

(b) From your answer in part (a), under what conditions on the data points (x_i, y_i), $i = 0, \ldots, 3$, we can have unique solution for c?

(c) Let the numerical values of the experimental data are $(0, 0); \quad (0.5, 4.25); \quad (1, 3); \quad (2, 2)$. From your answer in part (b), does unique c exist in this case? If yes, determine the vector c using any algorithm you have learned in this course.

(d) Determine the Lagrange interpolating polynomial that passes through the data points given in part (c).

(e) Compare the Lagrange interpolating polynomial derived in part (d), with the interpolating polynomial having monomial basis derived in part (c). Predict the value of y at $x = 1.5$ from part (c) and part (d).

(f) Submit a plot of the interpolating polynomial(s) in part (e). Mark the data points on your plot. (Hint: If using MATLAB, look up the commands `ezplot` and `plot` in MATLAB Help.

Also, you may find it useful to look at the commands in Homework 1, Problem 1 (d), for saving the MATLAB plot as .pdf file.)

Solution

(a) For interpolation, our cubic polynomial must pass through all the datapoints (x_i, y_i), $i = 0, \ldots, 3$. This gives four linear equations:

$$y_i = c_0 + c_1 x_i + c_2 x_i^2 + c_3 x_i^3, \qquad i = 0, 1, 2, 3,$$

which can be put in matrix-vector equation form

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

This matrix-vector equation needs to be solved for vector $c = \{c_0 \quad c_1 \quad c_2 \quad c_3\}^\top$.

(b) To solve for unique c, we need to have $\det \begin{vmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{vmatrix} \neq 0$. In other words, the determinant of the

Vandermonde matrix should be non-zero. One can prove (by induction) that the $n \times n$ Vandermonde determinant is (You don't need to prove this for the Homework. It is okay if you did something like this: `http://mathcentral.uregina.ca/QQ/database/QQ.09.07/h/rav1.html`):

$$\det \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \ldots & x_{n-1}^{n-1} \end{pmatrix} = \prod_{0 \leq i < j \leq n-1} (x_j - x_i).$$

In our case, the 4×4 Vandermonde determinant is equal to $(x_1 - x_0)(x_2 - x_0)(x_3 - x_0)(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$. Hence, there exists unique vector c provided all the abscissa values (x_i) of the experimental data, are distinct.

(c) Yes, for the numerical values of our experimental data, unique c exists since from part (b), the Vandermonde determinant $= 0.5 \times 1 \times 2 \times 0.5 \times 1.5 \times 1 = 0.75 \neq 0$.

To solve the matrix-vector equation in part (a), we first perform LU decomposition of the Vandermonde

2

matrix:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0.5 & 0.25 & 0.125 \\
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8
\end{pmatrix}
\xrightarrow[\text{Row 4=Row 4−Row 1, Row 3=Row 3−2Row 2}]{\text{Row 2=Row 2−Row 1, Row 3=Row 3−Row 1}}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\textcircled{1} & 0.5 & 0.25 & 0.125 \\
\textcircled{1} & \textcircled{2} & 0.5 & 0.75 \\
\textcircled{1} & 2 & 4 & 8
\end{pmatrix}
\xrightarrow[\text{Row 4=Row 4−6Row 3}]{\text{Row 4=Row 4−4Row 2}}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\textcircled{1} & 0.5 & 0.25 & 0.125 \\
\textcircled{1} & \textcircled{2} & 0.5 & 0.75 \\
\textcircled{1} & \textcircled{4} & \textcircled{6} & 3
\end{pmatrix}
\Rightarrow \quad
L =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 4 & 6 & 1
\end{pmatrix},
\quad
U =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0.5 & 0.25 & 0.125 \\
0 & 0 & 0.5 & 0.75 \\
0 & 0 & 0 & 3
\end{pmatrix}.
$$

Now we solve the matrix-vector equation in part (a) as $LUc = y$, where the vector $y = \{y_0 \quad y_1 \quad y_2 \quad y_3\}^\top$. To do this, we first solve for vector z in $Lz = y$, and then solve for vector c in $Uc = z$. Forward substitution in

$Lz = y$ yields $z = \begin{pmatrix} 0 \\ 4.25 \\ -5.5 \\ 18 \end{pmatrix}$. Then backward substitution in $Uc = z$ yields $c = \begin{pmatrix} 0 \\ 17 \\ -20 \\ 6 \end{pmatrix}$. Hence, the polynomial

that interpolates the given four points, is $6x^3 - 20x^2 + 17x$.

(d) The Lagrange interpolating polynomial is

$$
\sum_{i=0}^{3} y_i \, \ell_i(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2
$$

$$
+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3
$$

$$
= \frac{(x-0.5)(x-1)(x-2)}{(0-0.5)(0-1)(0-2)} \times 0 + \frac{(x-0)(x-1)(x-2)}{(0.5-0)(0.5-1)(0.5-2)} \times 4.25 + \frac{(x-0)(x-0.5)(x-2)}{(1-0)(1-0.5)(1-2)} \times 3
$$

$$
+ \frac{(x-0)(x-0.5)(x-1)}{(2-0)(2-0.5)(2-1)} \times 2
$$

$$
= 6x^3 - 20x^2 + 17x.
$$

(e) The polynomials computed from part (c) and (d) are the same. At $x = 1.5$, the predicted value of y is $6 \times (1.5)^3 - 20 \times (1.5)^2 + 17 \times 1.5 = 0.75$.
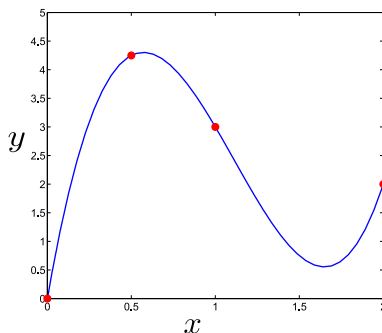
(f)

Figure 1: The *blue curve* is the interpolating polynomial $6x^3 - 20x^2 + 17x$ found in part (c) and (d), passing through the datapoints shown in *red circles*.

Problem 2

Interpolation using Neville's method and divided differences $(10+10+10 = 30$ **points)**

(a) For the experimental data given in Problem 1(c), construct the Neville table to interpolate the value of y at $x = 1.5$. How does your answer compare with that in Problem 1(e)?

(b) Repeat part (a) with the method of divided differences.

(c) If you are doing polynomial interpolation through a fixed set of $n + 1$ data points, is it possible that different algorithms (monomial basis, Lagrange polynomial, Neville's method, divided difference) can produce different answers? Why/why not?

Solution

(a) Neville's table can be constructed as shown below (see Example 3.2, page 155-156 in textbook, for details).

i	$|x - x_i|$	x_i	$y_i = P_{i0}$	P_{i1}	P_{i2}	P_{i3}
0	0.5	2	2	2.5	2.25	0.75
1	0.5	1	3	1.75	-3.75	
2	1	0.5	4.25	12.75		
3	1.5	0	0			

The answer is same as problem 1(e). Here is a nice online tool by Prof. Kiffe that you can use to verify your answer: `http://www.math.tamu.edu/~tkiffe/Fall12/609/Tools/interp.html`. Allow JAVA to run this on your web browser.

(b) We can construct the divided difference table as in page 159 in textbook. Again, this tells that the value of y at $x = 1.5$ is 0.75, as predicted by the solution of problem 1(e) and 2(a). You can use the same online tool, as above, to verify your results (select "Divided Differences" from the dropdown menu).

(c) No, it is *impossible* that different algorithms (monomial basis, Lagrange polynomial, Neville's method, divided difference) produce different interpolating polynomials.

The reason is that the nth degree polynomial that interpolates a fixed set of $n + 1$ data points, is unique. For a proof, see page 162 in textbook. Again, the proof is not mandatory for the purpose of this homework.

Problem 3

Least squares $(15 + 15 = 30$ **points)**

Consider the data in the following table.

x_i	1	2	4	5
y_i	2	3	5	6

We want to *approximate* the data using a quadratic polynomial $y = a_0 + a_1 x + a_2 x^2$ in least squares sense.

(a) Set up the matrix-vector equation to solve for the coefficient vector $a = \{a_0 \quad a_1 \quad a_2\}^\top$. Report the polynomial by solving the least squares problem. Submit a plot of the data points together with your least squares approximating polynomial. (Hint: You may use the LU decomposition technique to compute inverse of a square matrix, that you learned in Homework 4, Problem 1(f).)

(b) Approximate the same data using a straight line of the form $y = \alpha_0 + \alpha_1 x$, by resolving the least squares problem. Plot the approximating straight line on top of your plot in part (a). Report the straight line approximation and submit the final plot.

Solution

(a) Enforcing $y_i = a_0 + a_1 x_i + a_2 x_i^2$, $i = 0, \ldots, 3$, results the following system of linear equations:

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}}_{a} = \underbrace{\begin{pmatrix} 2 \\ 3 \\ 5 \\ 6 \end{pmatrix}}_{b}.$$

Unlike interpolation, the Vandermonde matrix X is rectangular and we can not exactly solve for the coefficient vector a from the equation $Xa = b$. The least square *approximation* for a is given by $\hat{a} = \left(X^\top X \right)^{-1} X^\top b$, which is computed as shown below.

First, we compute $X^\top b = \begin{pmatrix} 16 \\ 58 \\ 244 \end{pmatrix}$. To compute $\left(X^\top X \right)^{-1}$, we proceed similar to Problem 1(f) in Homework 4, as follows.

$$X^\top X = \begin{pmatrix} 4 & 12 & 46 \\ 12 & 46 & 198 \\ 46 & 198 & 898 \end{pmatrix} \xrightarrow[\text{Row 3=Row 3}-\frac{23}{2}\text{Row 1}]{\text{Row 2=Row 2}-3\text{Row 1}} \begin{pmatrix} 4 & 12 & 46 \\ ③ & 10 & 60 \\ \frac{23}{2} & 60 & 369 \end{pmatrix} \xrightarrow{\text{Row 3=Row 3}-6\text{Row 2}} \begin{pmatrix} 4 & 12 & 46 \\ ③ & 10 & 60 \\ \frac{23}{2} & ⑥ & 9 \end{pmatrix}$$

$$\Rightarrow \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ \frac{23}{2} & 6 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & 12 & 46 \\ 0 & 10 & 60 \\ 0 & 0 & 9 \end{pmatrix}.$$

Solving three linear systems $LUz_i = e_i$, $i = 1, 2, 3$, where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we obtain

$z_1 = \begin{pmatrix} \frac{1052}{180} \\ -\frac{139}{30} \\ \frac{13}{18} \end{pmatrix}$, $z_2 = \begin{pmatrix} -\frac{278}{60} \\ \frac{41}{10} \\ -\frac{2}{3} \end{pmatrix}$, $z_3 = \begin{pmatrix} \frac{13}{18} \\ -\frac{2}{3} \\ \frac{1}{9} \end{pmatrix}$. Stacking the vectors z_1, z_2, z_3 together, we get $\left(X^\top X\right)^{-1} =$

$\begin{pmatrix} \frac{1052}{180} & -\frac{278}{60} & \frac{13}{18} \\ -\frac{139}{30} & \frac{41}{10} & -\frac{2}{3} \\ \frac{13}{18} & -\frac{2}{3} & \frac{1}{9} \end{pmatrix}$. Then $\widehat{a} = \left(X^\top X\right)^{-1} X^\top b = \begin{pmatrix} \frac{1052}{180} & -\frac{278}{60} & \frac{13}{18} \\ -\frac{139}{30} & \frac{41}{10} & -\frac{2}{3} \\ \frac{13}{18} & -\frac{2}{3} & \frac{1}{9} \end{pmatrix} \begin{pmatrix} 16 \\ 58 \\ 244 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Hence, the least

square quadratic polynomial is $y = 1 + x$. This is an interesting result since the best quadratic approximation turns out to be a linear approximation! If we look at our data, we can explain why this strange thing has happened: all our datapoints were indeed collinear!

(b) In this case, we have

$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}}_{X_{\text{new}}} \underbrace{\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}}_{\alpha} = \underbrace{\begin{pmatrix} 2 \\ 3 \\ 5 \\ 6 \end{pmatrix}}_{b} \quad \Rightarrow \quad X_{\text{new}}^\top X_{\text{new}} = \begin{pmatrix} 4 & 12 \\ 12 & 46 \end{pmatrix} \quad \Rightarrow \quad \left(X_{\text{new}}^\top X_{\text{new}}\right)^{-1} = \frac{1}{(4 \times 46) - (12 \times 12)} \begin{pmatrix} 46 & -12 \\ -12 & 4 \end{pmatrix}$

$$= \frac{1}{40} \begin{pmatrix} 46 & -12 \\ -12 & 4 \end{pmatrix} = \begin{pmatrix} \frac{23}{20} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{pmatrix}.$$

On the other hand, $X_{\text{new}}^\top b = \begin{pmatrix} 16 \\ 58 \end{pmatrix}$. Thus, we have $\widehat{\alpha} = \left(X_{\text{new}}^\top X_{\text{new}}\right)^{-1} X_{\text{new}}^\top b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence, the best linear

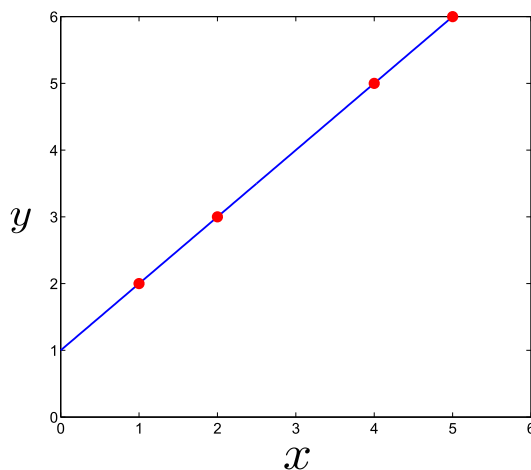approximation is $y = 1 + x$, which is same as the answer we got in part (a). The plot is shown below.

Figure 2: The *blue curve* is the least square polynomial $1 + x$ found in part (a) and (b). The datapoints are shown as *red circles*.