Aero 320: Numerical Methods

Homework 7

Name: ....

Due: November 27, 2013

NOTE: All problems, unless explicitly asked to write a code, are to be done by hand (with the help of a calculator) but **you need to show all the steps**. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Submit your HW by Wednesday midnight at Room 201, Reed McDonald Building. Late submissions or failure to submit in the required format will receive no credit.

Problem 1

Application of numerical differentiation: heat equation ($10+10+10+5=35$ **points)**

Please review Lab 17 before you start this problem. Consider the numerical approximation of the heat equation that we derived in Lab 17, Problem 1(c). In this exercise, we will numerically solve that equation. To do this, first notice that the boundary conditions can be written in discrete form:

$$\begin{aligned} u_{i,0} &= \phi\left(x_i\right), \qquad && i = 0, 1, \ldots, M, \\ u_{0,k} = u_{M,k} &= 0, \qquad && k = 1, 2, \ldots, N. \end{aligned}$$

(a) Using the above information, rewrite the approximated heat equation derived in Lab 17, Problem 1(c), in matrix-vector form

$$\begin{pmatrix} u_{1,k+1} \\ u_{2,k+1} \\ \vdots \\ u_{M-1,k+1} \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 & 0 & \ldots & 0 \\ b & a & b & 0 & 0 & \ldots & 0 \\ 0 & b & a & b & 0 & \ldots & 0 \\ 0 & 0 & b & a & b & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & b & a \end{pmatrix} \begin{pmatrix} u_{1,k} \\ u_{2,k} \\ \vdots \\ u_{M-1,k} \end{pmatrix}.$$

Find a and b in the above matrix, in terms of λ (we defined λ in Lab 17).

(b) Assume $\alpha = 1$, $L = 1$, $\Delta x = 0.1$, $\Delta t = 0.001$, $T = 1$. (There was a typo in Lab 17:

Δk in page 1 should be Δt) The attached MATLAB code `HeatEqnFTCS.m` iteratively solves the FTCS equation derived in Lab 17, Problem 1(c), assuming the heat distribution at time $t = 0$ as $u(x, 0) = \phi(x) = \sin\left(\frac{\pi x}{L}\right)$. The exact analytical solution is $u(x, t) = \sin\left(\frac{\pi x}{L}\right) \exp\left(-\frac{\alpha \pi^2 t}{L^2}\right)$. Run the MATLAB code that plots both analytical and exact solutions for different time instants. At any time, which location of the rod is most hot? Give intuitive reasons in support of your answer.

(c) What happens to the temperature distribution along the rod, in general, as time passes by? What will happen to the temperature distribution if we wait long time ($T \to \infty$)? Explain.

(d) For our derivative approximations in the heat equation, we had $O(\Delta t)$ error for time derivative, and $O\left((\Delta x)^2\right)$ error for spatial derivative. What kind of errors are these? Instead of *forward difference in time, central difference in space (FTCS)* approximation, if we instead use *backward difference in time, central difference in space (BTCS)* approximation, then will your order of total error change? Why/why not?

Problem 2

More on numerical differentiation **($3+5+5+10+(2+2+2+1) = 30$ points)**

Consider $f(x) = \ln x$. We want to compute $f'(x)$ at $x = 3$, using different numerical methods.

(a) Find the exact expression for $\frac{d}{dx} \ln x$. Evaluate your answer numerically at $x = 3$ to find $f'_{\text{exact}}(3)$. Keep up to 8 significant digits.

(b) Compute the *second order central difference* approximation $f'_{h_1}(3)$ with step size $h_1 = 0.4$. Keep up to 8 significant digits.

(c) Compute the *second order central difference* approximation $f'_{h_2}(3)$ with step size $h_2 = \frac{h_1}{2} = 0.2$. Keep up to 8 significant digits.

(d) Using your answers in part (b) and (c), compute the (second order) *Richardson extrapolation* $f'_{\text{Richardson}}(3)$. Keep up to 8 significant digits.

(e) Consider your answer in part (a) to be the true/exact value of the derivative at $x = 3$. Compute the *absolute* and *relative errors* in the answers of part (b), (c) and (d). Which one seems to be the best algorithm to numerically compute the derivative?

Problem 3

Numerical integration **(**$5+5+5+5+(2+2+2+2+1+6) = 35$ **points)**

Let $f(x) = \dfrac{1}{1+x^2}$. In this exercise, we want to numerically compute $\displaystyle\int_{-1}^{1} f(x)\ dx$.

(a) Compute the integral using *three point Simpson's method.*

(b) Compute the integral using *five point Simpson's method.*

(c) Partition the interval $[-1, 1]$ into 2 subintervals: $[-1, 0]$ and $[0, 1]$. For this partition, compute the integral using *trapezoid method.*

(d) Using the same partition as in part (c), compute the integral using *midpoint method.*

(e) The exact value of our integral is $\dfrac{\pi}{2}$. Compute the *absolute* and *relative errors* in your answers in part (a), (b), (c), (d). Among the methods in (a), (b), (c), (d), which method performs better than which? Why?