Aero 320: Numerical Methods

Lab Assignment 14

Fall 2013

Problem 1

Spline interpolation

(a) Generate a table of datapoints (x_i, y_i), $i = 0, \ldots, 5$, by sampling the function $y = \dfrac{1}{1+25x^2}$. To do this, choose $x_i = -1 : 0.4 : 1$. Then evaluate these x_i exactly at the function to determine y_i.

(b) Set up the matrix-vector equations needed to interpolate these data via linear spline. Repeat the same for quadratic and cubic splines.

(c) Solve the system of linear equations (by hand OR by writing a code) using algorithms you learned in this course (for example, LU decomposition or Gauss elimination).

(d) Plot your spline interpolations together with the datapoints. Compare your results with the MATLAB plots (shown in lab).

Solution

(a)

x_i	y_i
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

(b) Since there are six datapoints, we have five intervals, and hence five polynomials to find. For linear spline, we need to find polynomials of the form $y = a_i x + b_i$, where $i = 1, \ldots, 5$. We impose continuity at the knot points to derive

$$\begin{aligned} y_0 &= a_1 x_0 + b_1, \\ y_1 &= a_1 x_1 + b_1, \\ y_1 &= a_2 x_1 + b_2, \\ y_2 &= a_2 x_2 + b_2, \\ &\vdots \\ y_5 &= a_5 x_5 + b_5. \end{aligned}$$

This results a matrix-vector equation with block matrix structure, as shown below, to be solved for the column vector $\{a_1, b_1, a_2, b_2, \ldots, a_5, b_5\}^\top$.

$$\begin{pmatrix} x_0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ x_1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & x_1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & x_2 & 1 & 0 & 0 & \ldots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & x_5 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_5 \\ b_5 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_1 \\ y_2 \\ \vdots \\ y_4 \\ y_5 \end{pmatrix}.$$

The matrix-vector equation for quadratic spline (where each of the five polynomials have the form $y = a_i x^2 + b_i x + c_i$, $i = 1, \ldots, 5$) has a similar block structure, given in Lecture slide #5 in `320-ch3-b.pdf`. I give here some details, as it may help you understand better.

There are total $(n+1)$ datapoints, and n intervals. In spline interpolation, since one polynomial needs to be fitted per interval, hence there are total n polynomials. In quadratic spline, there are 3 coefficients (a_i, b_i, c_i) for each polynomial, and therefore, total $3n$ unknown coefficients.

Now let's calculate how many equations we've got. In the data, there are 2 endpoints and $(n + 1 - 2) = (n - 1)$ interior points. Every interior point generates 3 equations (2 for continuity and 1 for matching slope), thus total $3\,(n-1)$ equations come from the interior points. The two endpoints generate 3 equations (2 for continuity and 1 extra equation by setting the left most spline to be linear, i.e. $a_1 = 0$). Hence there are total $3\,(n-1) + 3 = 3n$ equations available to solve for $3n$ unknown coefficients. Thus, we must have a matrix-vector equation involving a square matrix (unlike least squares, where we had a rectangular matrix). The equations are given below.

$$
\begin{aligned}
0 &= a_1, \\
y_0 &= a_1 x_0^2 + b_1 x_0 + c_1, \\
y_1 &= a_1 x_1^2 + b_1 x_1 + c_1, \\
y_1 &= a_2 x_1^2 + b_2 x_1 + c_2, \\
0 &= 2\left(a_1 - a_2\right) x_1 + \left(b_1 - b_2\right), \\
y_2 &= a_2 x_2^2 + b_2 x_2 + c_2, \\
y_2 &= a_3 x_2^2 + b_3 x_2 + c_3, \\
0 &= 2\left(a_2 - a_3\right) x_2 + \left(b_2 - b_3\right), \\
y_3 &= a_3 x_3^2 + b_3 x_3 + c_3, \\
&\vdots \\
y_5 &= a_5 x_5^2 + b_5 x_5 + c_5.
\end{aligned}
$$

The matrix-vector form becomes

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_0^2 & x_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1^2 & x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1^2 & x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2x_1 & 1 & 0 & -2x_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_2^2 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2^2 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2x_2 & 1 & 0 & -2x_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_3^2 & x_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3^2 & x_3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2x_3 & 1 & 0 & -2x_3 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4^2 & x_4 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4^2 & x_4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2x_4 & 1 & 0 & -2x_4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5^2 & x_5 & 1
\end{pmatrix}
\begin{pmatrix}
a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5
\end{pmatrix}
=
\begin{pmatrix}
0 \\ y_0 \\ y_1 \\ y_1 \\ 0 \\ y_2 \\ y_2 \\ 0 \\ y_3 \\ y_3 \\ 0 \\ y_4 \\ y_4 \\ 0 \\ y_5
\end{pmatrix}.
$$

The matrix-vector equation for cubic spline case will be done in Homework 6.

(c) For the linear case, it is easy to solve for the coefficients by hand. The solution is given in Lecture slide #2 (see bottom right corner) in `320-ch3-b.pdf`, which is

$$
a_1 = \frac{y_1 - y_0}{x_1 - x_0}, \qquad b_1 = y_0 - x_0 \left(\frac{y_1 - y_0}{x_1 - x_0}\right), \text{etc.}
$$

In general, we get $a_i = \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}$, and $b_i = y_{i-1} - x_{i-1}\left(\dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}\right)$, $i = 1, \ldots, 5$. The matrix-vector equations

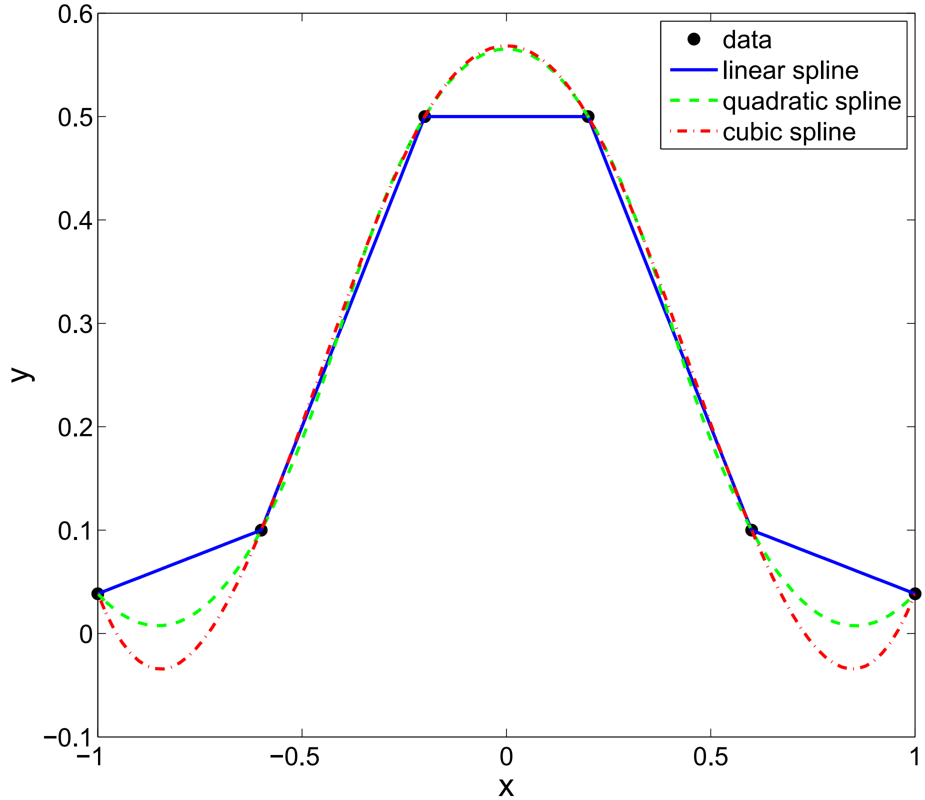

Figure 1: The MATLAB spline interpolations computed from `SplineExample.m`.

for quadratic and cubic case, can be solved by the Gauss elimination or LU decomposition codes sent earlier.

(d) MATLAB results are generated from `SplineExample.m` (attached). To generate the same from C++ code, see the .cpp file sent as solution for Homework 6, Problem 2(c).