Aero 320: Numerical Methods

Lab Assignment 16

Fall 2013

Problem 1

Numerical differentiation

For small h, where $0 < h << 1$, use Taylor series expansion to show that
(a)

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + O(h),$$

(b)

$$f''(x_0) = \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2} + O\left(h^2\right).$$

Solution

(a) Taylor series expansion of the function f about x_0 gives

$$\begin{aligned} f(x_0 + h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \dots \\ \Rightarrow f'(x_0) &= \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) - \dots \\ &= \frac{f(x_0 + h) - f(x_0)}{h} + O(h). \end{aligned}$$

This is the *two point forward difference* approximation for the first derivative.
(b) From Taylor series expansion similar to part (a), we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \dots$$

and (think of this as replacing h by $-h$ in the above expansion)

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \dots$$

Adding the above two equations, we get

$$
\begin{aligned}
f\left(x_0+h\right)+f\left(x_0-h\right) &= 2f\left(x_0\right)+h^2 f''\left(x_0\right)+\frac{h^4}{12}f''''\left(x_0\right)+\ldots \\
\Rightarrow f\left(x_0+h\right)-2f\left(x_0\right)+f\left(x_0-h\right) &= h^2 f''\left(x_0\right)+\frac{h^4}{12}f''''\left(x_0\right)+\ldots \\
\Rightarrow f''(x_0) &= \frac{f\left(x_0-h\right)-2f\left(x_0\right)+f\left(x_0+h\right)}{h^2} - \frac{h^2}{12}f''''\left(x_0\right)-\ldots \\
&= \frac{f\left(x_0-h\right)-2f\left(x_0\right)+f\left(x_0+h\right)}{h^2} + O\left(h^2\right).
\end{aligned}
$$

This is the *three point central difference* approximation for the second derivative.