Aero 320: Numerical Methods

Lab Assignment 17

Fall 2013

Problem 1

Application of numerical differentiation: solving Partial Differential Equations (PDEs)

Consider the one dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad 0 < t < T,$$

that describes the evolution of temperature $u(x,t)$ along a bar of length L, as a function of position x and time t. Here α denotes the thermal diffusivity of the material. Let us assume the following boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad 0 < t < T,$$
$$u(x,0) = \phi(x), \qquad 0 \leq x \leq L.$$

Partition the space interval $[0, L]$ as

$$x_i = i\Delta x, \qquad i = 0, 1, \ldots, M, \qquad \Delta x = \frac{L}{M}.$$

Similarly partition the time interval $[0, T]$ as

$$t_k = k\Delta t, \qquad k = 0, 1, \ldots, N, \qquad \Delta k = \frac{T}{N}.$$

Let us introduce the notation $u_{i,k} = u(x_i, t_k)$.

(a) Write the *two point forward difference* approximation for the left hand side time derivative of the heat equation.

(b) Write the *three point central difference* approximation for the right hand side spatial derivative of the heat equation.

(c) Use your answer from part (a) and (b), to approximate the heat equation as

$$u_{i,k+1} = (1 - 2\lambda)\, u_{i,k} + \lambda\,(u_{i+1,k} + u_{i-1,k}), \qquad \text{where } \lambda = \frac{\alpha^2 \Delta t}{(\Delta x)^2}.$$

Solution

(a) The two point forward difference approximation for first order derivative with respect to time, gives

$$\frac{\partial u}{\partial t} \approx \frac{u_{i,k+1} - u_{i,k}}{\Delta t}.$$

(b) The three point central difference approximation for second order derivative with respect to space, gives

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,k} - 2u_{i,k} + u_{i-1,k}}{(\Delta x)^2}.$$

(c) Substituting the approximations from part (a) and (b) in the heat equation, we get

$$\frac{u_{i,k+1} - u_{i,k}}{\Delta t} = \alpha^2 \frac{u_{i+1,k} - 2u_{i,k} + u_{i-1,k}}{(\Delta x)^2}$$

$$\Rightarrow u_{i,k+1} - u_{i,k} = \lambda \left(u_{i+1,k} - 2u_{i,k} + u_{i-1,k}\right)$$

$$\Rightarrow u_{i,k+1} = (1 - 2\lambda)\, u_{i,k} + \lambda \left(u_{i+1,k} + u_{i-1,k}\right), \qquad \text{where } \lambda = \frac{\alpha^2 \Delta t}{(\Delta x)^2}.$$

The above formula is often called FTCS (forward in time, central in space) approximation.