A Proximal Algorithm for Uncertainty Propagation in Stochastic Nonlinear Systems

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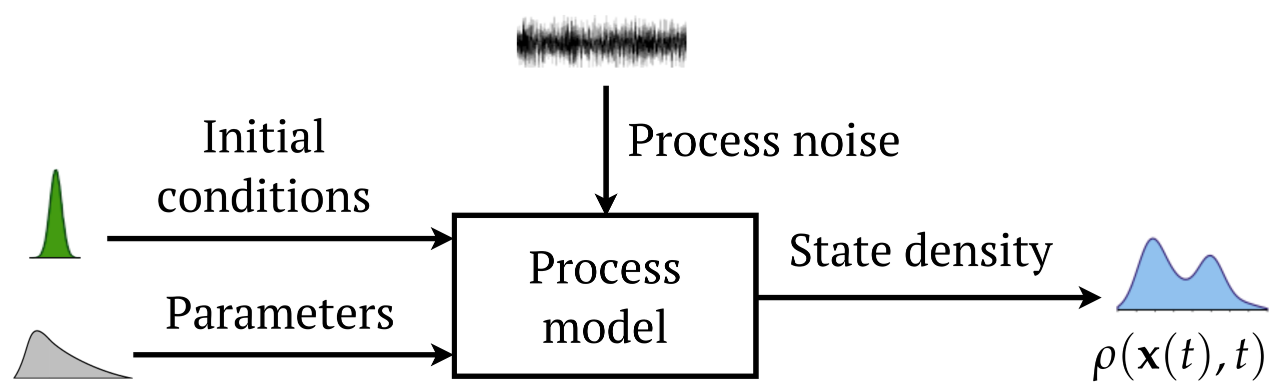
Problem: Uncertainty Propagation



Trajectory flow (Itô SDE):

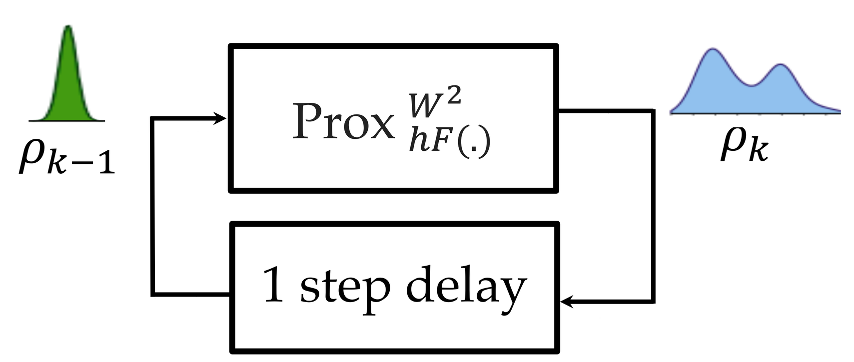
$$\mathrm{d}\boldsymbol{x}(t) = -\nabla\psi(\boldsymbol{x},t)\,\mathrm{d}t + 2\beta^{-1}\,\mathrm{d}\boldsymbol{w}(t), \qquad \boldsymbol{x}\in\mathbb{R}^n,$$

$$\boldsymbol{x}(0)\sim\rho_0(\boldsymbol{x}), \qquad \mathrm{d}\boldsymbol{w}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{I}\mathrm{d}t)$$

Joint PDF flow (Fokker-Planck-Kolmogorov PDE):

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\nabla\psi) + \beta^{-1}\Delta\rho, \qquad \rho(\boldsymbol{x},t=0) = \rho_0(\boldsymbol{x})$$

Main Idea: Variational Recursion



Proximal recursion:

$$\rho_k = \text{Prox}^{W^2}_{hF(.)}(\rho_{k-1}) := \underset{\rho}{\arg\inf}\ \frac{1}{2}W^2(\rho_{k-1},\rho) + h\,F(\rho)$$

Monge-Kantorovich optimal transport cost:

$$W^2(\rho_{k-1},\rho) := \inf_{\Phi\in\Pi(\rho_{k-1},\rho)} \int c(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\Phi(\boldsymbol{x},\boldsymbol{y})$$

Free energy functional:

$$F(\rho) := \int \psi\,\rho\,\mathrm{d}\boldsymbol{x} + \beta^{-1}\int\rho\log\rho\,\mathrm{d}\boldsymbol{x}$$

Algorithm: Gradient Ascent on the Dual Space

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\nabla\psi) + \beta^{-1}\Delta\rho$$

$\Updownarrow$ JKO variational scheme, 1998

$$\rho_k = \rho(\boldsymbol{x}, t=kh) = \underset{\rho}{\arg\inf}\ \frac{1}{2}W^2(\rho_{k-1},\rho) + h\,F(\rho),\ h \text{ small}$$

$\Downarrow$ Discrete primal formulation

$$\boldsymbol{\rho}_k = \underset{\boldsymbol{\rho}}{\arg\min}\left\{\min_{\boldsymbol{\Phi}\in\Pi(\boldsymbol{\rho}_{k-1},\boldsymbol{\rho})}\frac{1}{2}\langle\boldsymbol{C},\boldsymbol{\Phi}\rangle + h\,\langle\boldsymbol{\psi}+\log\boldsymbol{\rho},\boldsymbol{\rho}\rangle\right\}$$

$\Downarrow$ Entropic regularization

$$\boldsymbol{\rho}_k = \underset{\boldsymbol{\rho}}{\arg\min}\left\{\min_{\boldsymbol{\Phi}\in\Pi(\boldsymbol{\rho}_{k-1},\boldsymbol{\rho})}\left\{\frac{1}{2}\langle\boldsymbol{C},\boldsymbol{\Phi}\rangle + \epsilon\,D(\boldsymbol{\Phi})\right\} + h\,\langle\boldsymbol{\psi}+\log\boldsymbol{\rho},\boldsymbol{\rho}\rangle\right\}$$

$\Downarrow$ Dualization

$$\boldsymbol{\lambda}_0^*,\boldsymbol{\lambda}_1^* = \underset{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1}{\arg\max}\left\{\frac{1}{2}\langle\boldsymbol{\lambda}_0,\boldsymbol{\rho}_{k-1}\rangle - F^*(-\boldsymbol{\lambda}_1) - \epsilon\left(e^{\frac{\boldsymbol{\lambda}_0}{\epsilon}}\right)^\top\left(e^{\frac{-\boldsymbol{C}}{\epsilon}}\right)\left(e^{\frac{\boldsymbol{\lambda}_1}{\epsilon}}\right)\right\}$$

$$\boldsymbol{y} = e^{\frac{\boldsymbol{\lambda}_0^*}{\epsilon}} \ \Big\downarrow \quad \Big\downarrow\ \boldsymbol{z} = e^{\frac{\boldsymbol{\lambda}_1^*}{\epsilon}}$$

Coupled Transcendental Equations in y and z

Inputs: $\boldsymbol{\Gamma} = e^{\frac{-\boldsymbol{C}}{\epsilon}}$, $\boldsymbol{\rho}_{k-1}$, $\boldsymbol{\xi} = \dfrac{e^{-\beta\boldsymbol{\psi}}}{e}$ $\longrightarrow$

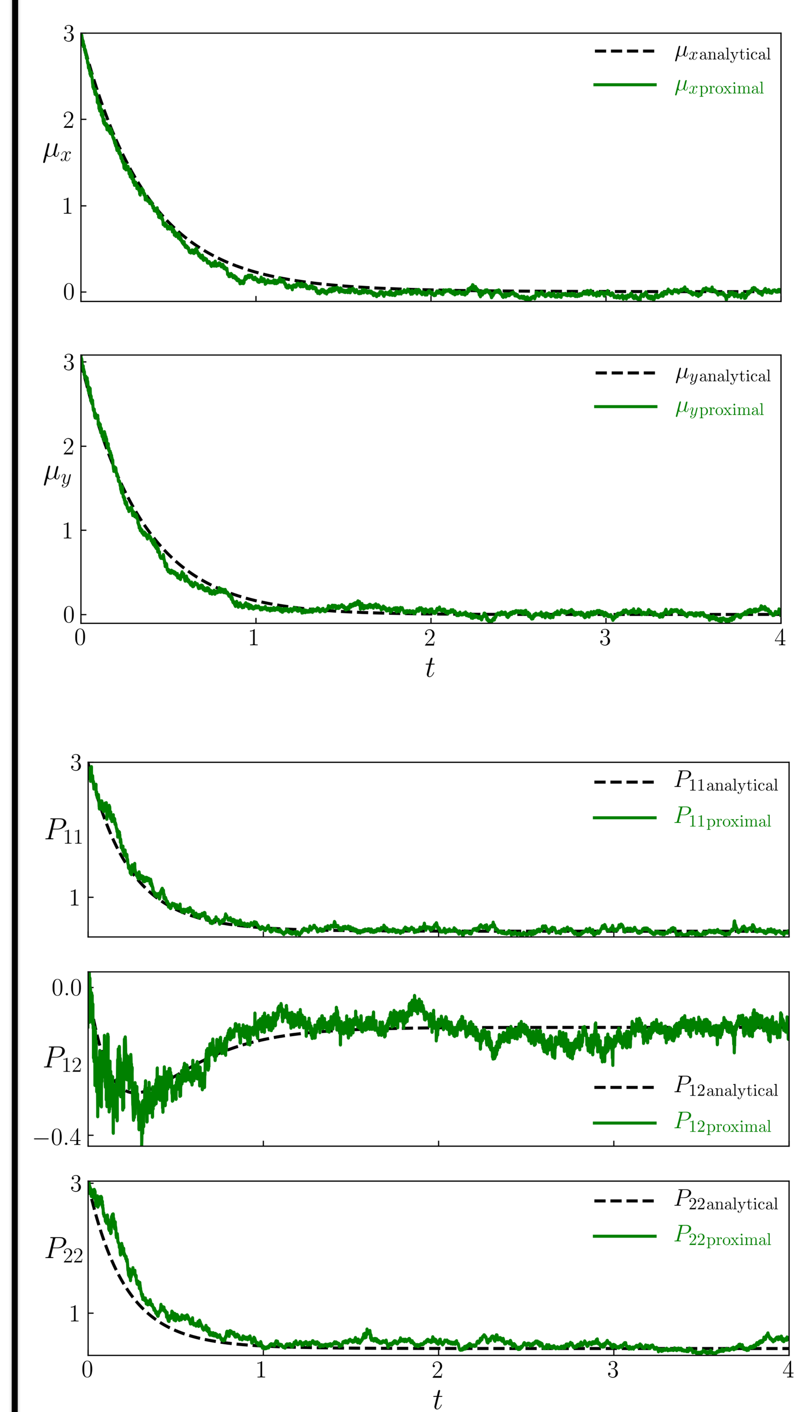
$$\boldsymbol{y}\odot\boldsymbol{\Gamma}\boldsymbol{z} = \boldsymbol{\rho}_{k-1}$$

$$\boldsymbol{z}\odot\boldsymbol{\Gamma}^\top\boldsymbol{y} = \boldsymbol{\xi}\odot\boldsymbol{z}^{-\beta\epsilon/2h}$$

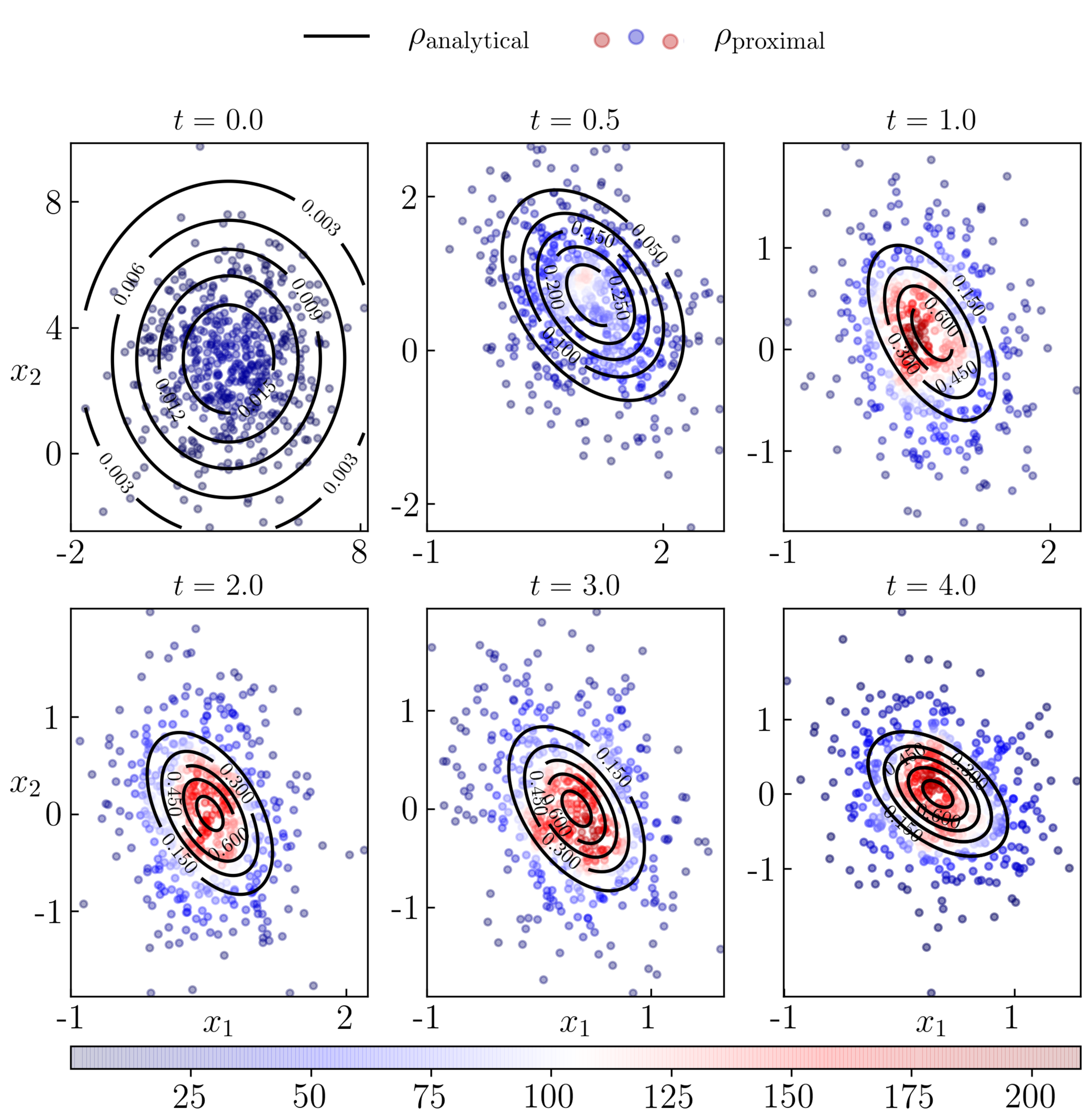
$\longrightarrow\ \boldsymbol{\rho}_k = \boldsymbol{z}\odot\boldsymbol{\Gamma}^\top\boldsymbol{y}$

> **Theorem**
> The fixed point iteration is strictly contractive with respect to the Thompson Metric. Thus it admits a unique fixed point.

Ornstein-Uhlenbeck SDE in Two Dimensions:

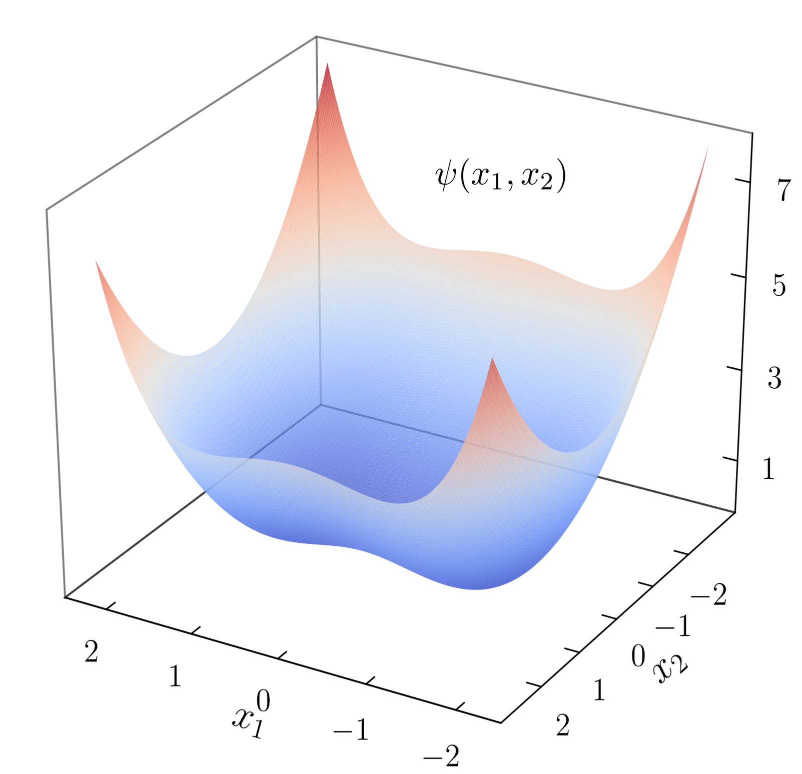


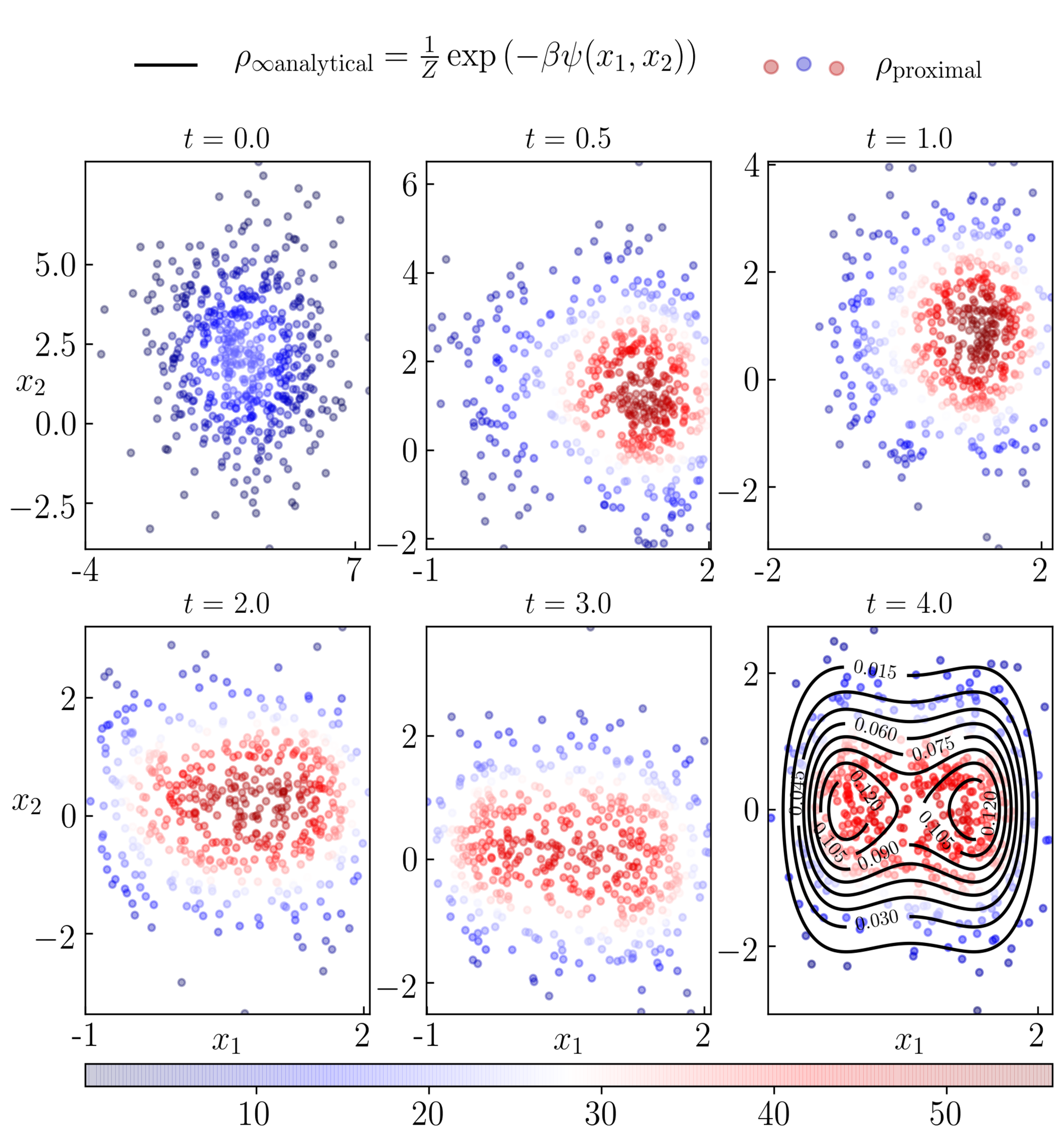
$$\psi(x) = \frac{1}{2}x^\top A x, \quad A\succ 0$$



Nonlinear SDE with Gradient Drift Field in Two Dimensions:

$$\psi(x_1,x_2) = \frac{1}{4}\left(1+x_1^4\right) + \frac{1}{2}\left(x_2^2 - x_1^2\right)$$





Numerical Simulation

Ornstein-Uhlenbeck SDE in One Dimension:

$$\psi(x) = \frac{1}{2}ax^2, \quad a>0$$