# Uncertainties in Control Systems

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# Recap: modeling control systems

All our analysis, MATLAB simulation, control design used standard state space form

In practice, we often encounter difference equations with time delay/memory

Difference equations can be brought to state space forms by re-writing

# Recap: discrete time control systems in state space form

**Example (linear control system):** two process states $(x_1, x_2)$ and one control $u$

$$x_1(t+1) = a_{11}x_1(t) + a_{12}x_2(t) + b_{11}u(t)$$
$$x_2(t+1) = a_{21}x_1(t) + a_{22}x_2(t) + b_{21}u(t)$$

where the coefficients $a'$s and $b'$s are known constants

**Example (nonlinear control system):** three process states $(x_1, x_2, \theta)$ and two controls $(V, \omega)$

$$x_1(t+1) = x_1(t) + V(t)\Delta t \times \cos\theta(t)$$
$$x_2(t+1) = x_2(t) + V(t)\Delta t \times \sin\theta(t)$$
$$\theta(t+1) = \theta(t) + \Delta t \times \omega(t) + w(t)$$

# Recap: write the following in state space form

$$x(t+1) + 2x(t) - 5x(t-1) + 7x(t-2) = 3u(t)$$

$$y(t) = 4x(t) + 5x(t-1)$$

**Solution:** introduce new variables:

$$x_1(t) := x(t)$$
$$x_2(t) := x(t-1) = x_1(t-1)$$
$$x_3(t) := x(t-2) = x_2(t-1)$$

Therefore

$$x_1(t+1) = -2x_1(t) + 5x_2(t) - 7x_3(t) + 3u(t)$$
$$x_2(t+1) = x_1(t)$$
$$x_3(t+1) = x_2(t)$$

sensor / measurement model

$$y(t) = 4x_1(t) + 5x_2(t)$$

**MATLAB exercise: controllable or not?**

# General state space form: without noise

**Process model**:

$$x_1(t+1) = f_1\left(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)\right)$$
$$x_2(t+1) = f_2\left(x_2(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)\right)$$
$$\vdots = \vdots$$
$$x_n(t+1) = f_n\left(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)\right)$$

**Measurement model**:

$$y_1(t) = g_1\left(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)\right)$$
$$y_2(t) = g_2\left(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)\right)$$
$$\vdots = \vdots$$
$$y_p(t) = g_p\left(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)\right)$$

$n$ **states** $x_1, \ldots, x_n$

$m$ **controls** $u_1, \ldots, u_m$

$p$ **outputs** $y_1, \ldots, y_p$

# General state space form: with noise

**Process model**:

$$x_1(t+1) = f_1\left(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), w_1(t), \ldots, w_q(t)\right)$$

$$x_2(t+1) = f_2\left(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), w_1(t), \ldots, w_q(t)\right)$$

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$p$ **outputs** $y_1, \ldots, y_p$

$q$ **process noises** $w_1, \ldots, w_q$

$r$ **measurement noises** $v_1, \ldots, v_r$

**Measurement model**:

$$y_1(t) = g_1\left(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), v_1(t), \ldots, v_r(t)\right)$$

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**Measurement model**:

$$y_1(t) = g_1\left(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), v_1(t), \ldots, v_r(t)\right)$$

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$n$ **states** $x_1, \ldots, x_n$

$m$ **controls** $u_1, \ldots, u_m$

$p$ **outputs** $y_1, \ldots, y_p$

$q$ **process noises** $w_1, \ldots, w_q$

$r$ **measurement noises** $v_1, \ldots, v_r$

**In addition, there could be (static) parameters** $p_1, \ldots, p_s$

In simple pendulum, what are the parameters and states?

# Sources of uncertainties in control systems

**Initial conditions:** $x_1(0), \ldots, x_n(0)$ may not be exactly known

**Parameters:** $p_1, \ldots, p_s$ may not be exactly known

**Noises:** $w_1, \ldots, w_q$ and $v_1, \ldots, v_r$ if present, are unknown and unmeasured

Question: What do the noises actually represent?

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Question: What do the noises actually represent?

Answer: (i) external disturbance (example: wind gust in an airplane)
(ii) unmodeled process dynamics (example: unknown or complicated physics)

# Types of uncertainties

**One way to classify:** epistemic/structured versus aleatoric/unstructured uncertainties

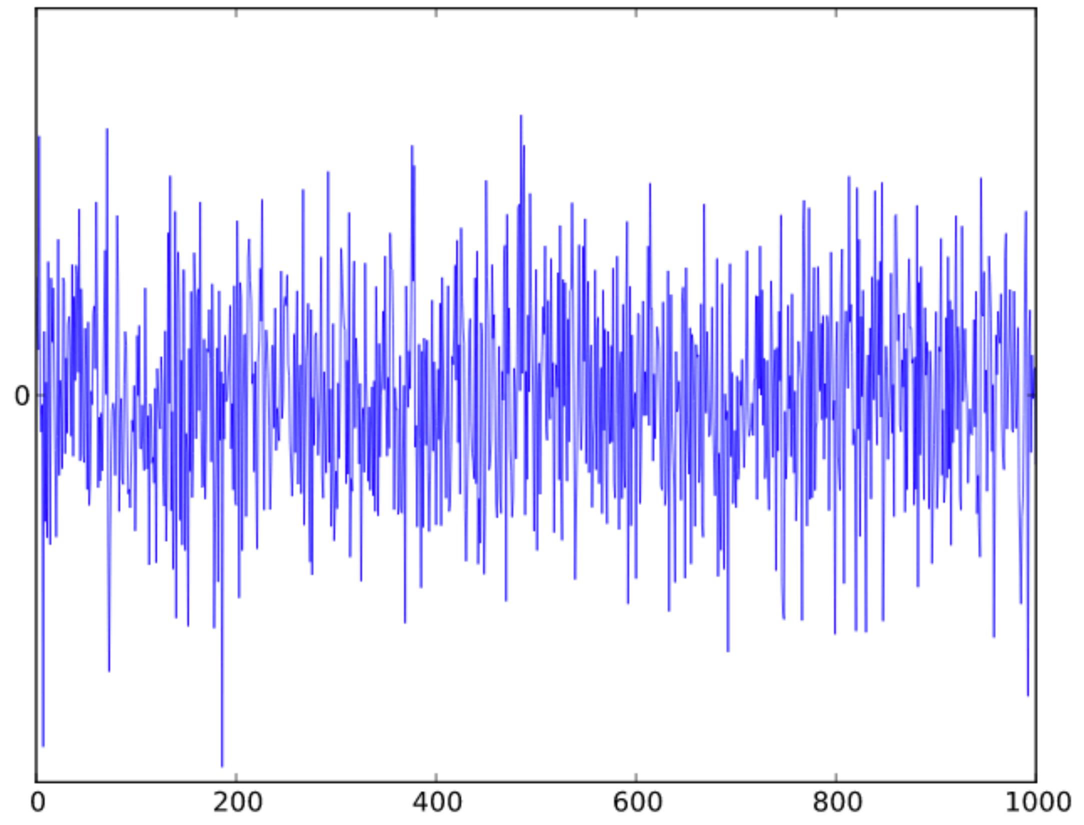
**Meanings of these nomenclature**

**Examples:** $x(t) = a \sin(bt)$ where $a, b$ unknown

$x(t) = \sin(t) +$ random white noise

**Another way to classify:** deterministic versus stochastic/probabilisitic uncertainties

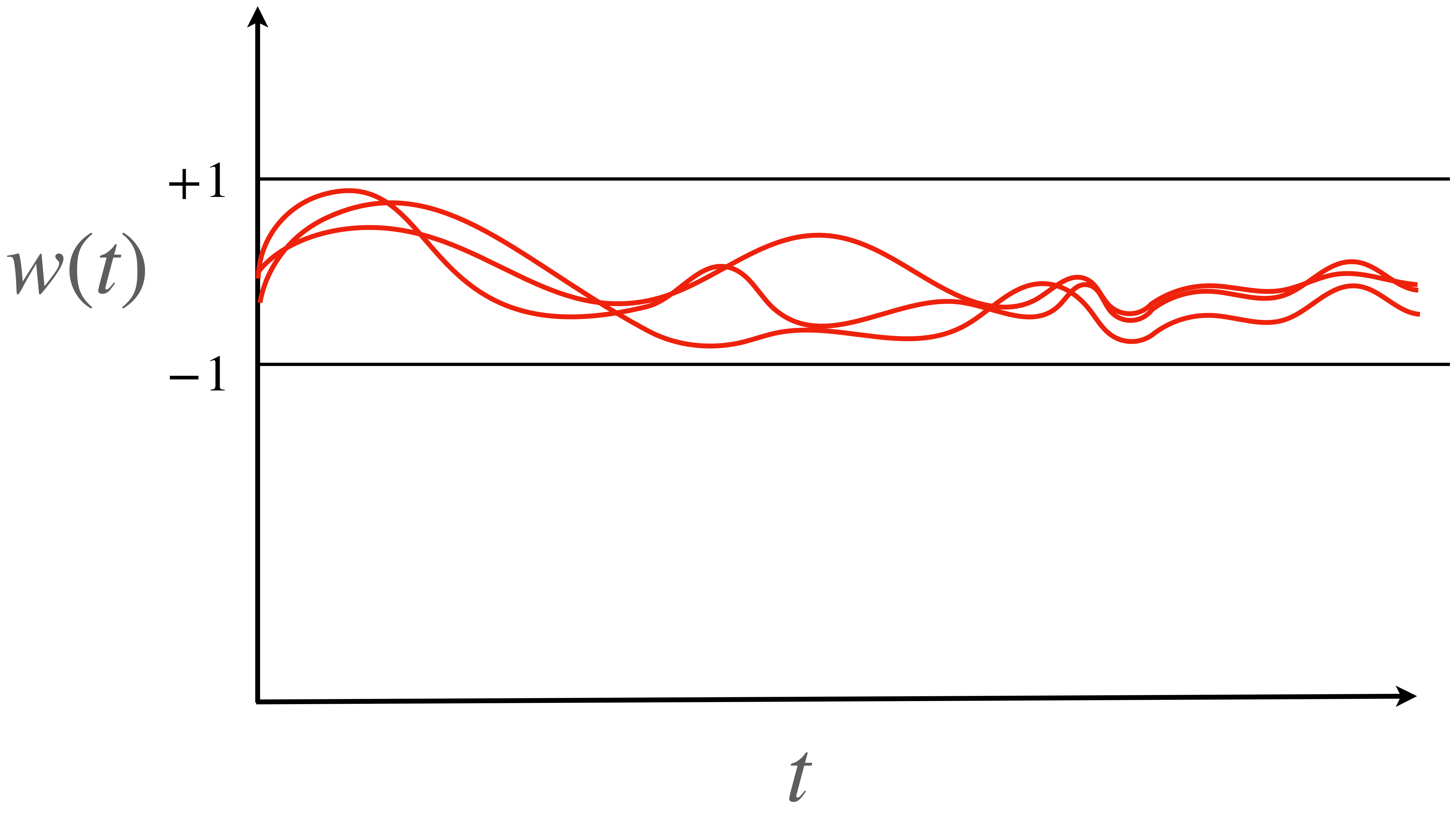
**Examples:** $x(t+1) = 2x(t) + w(t)$ where $w(t)$ is a Gaussian white noise




$x(t+1) = 2x(t) + w(t)$ where $-1 \le w(t) \le 1$

# Deterministic uncertainties

Deterministic uncertainties ≡ set valued uncertainties




In the above plot, the set = interval $[-1,1]$

If $-1 \leq w_1(t) \leq +1, \quad 2 \leq w_2(t) \leq 3, \quad 0 \leq w_3(t) \leq 4$, then the set is

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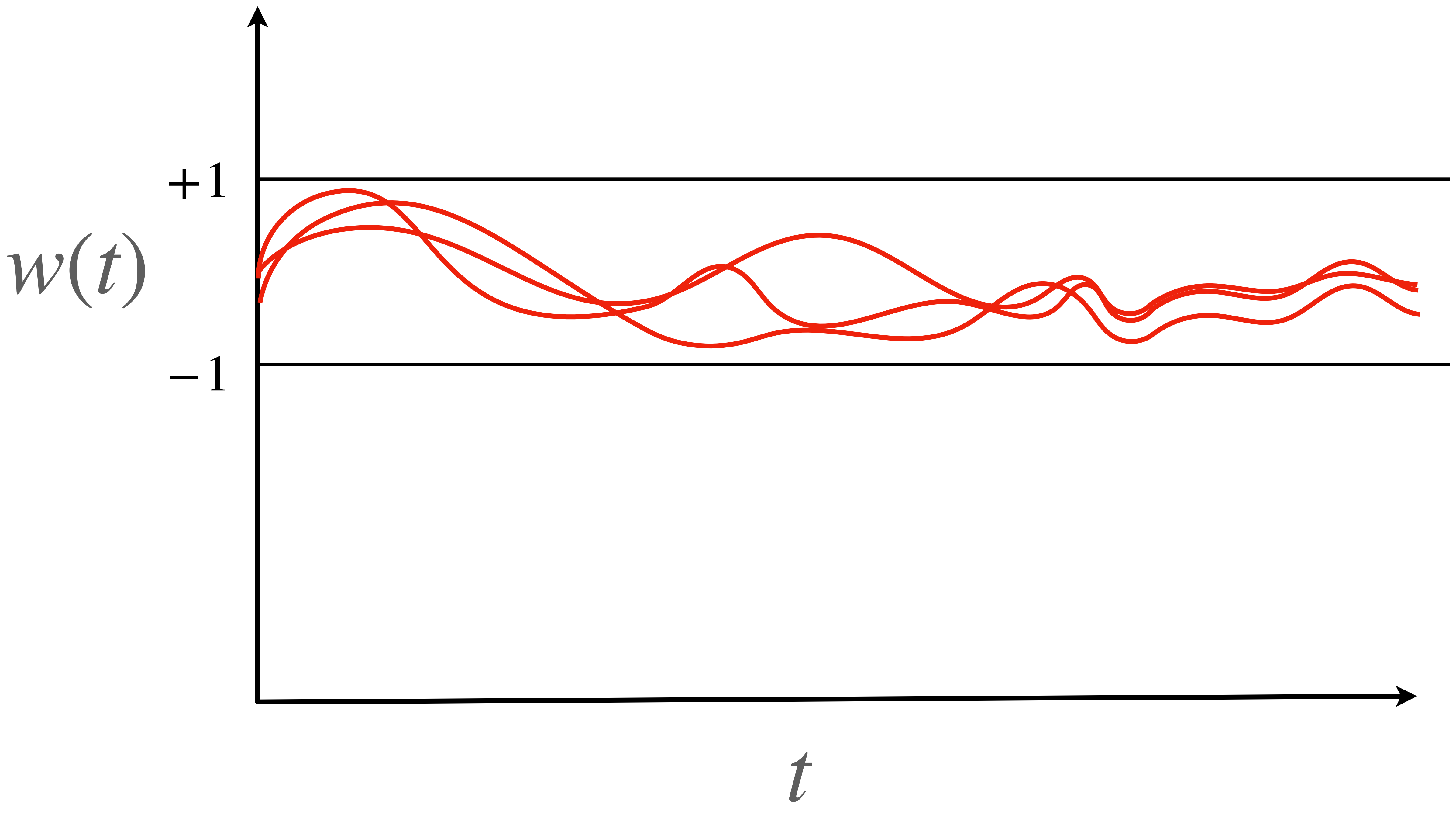


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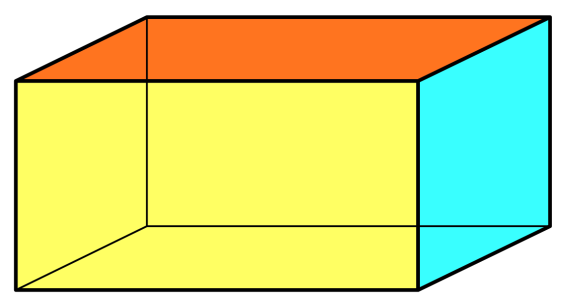

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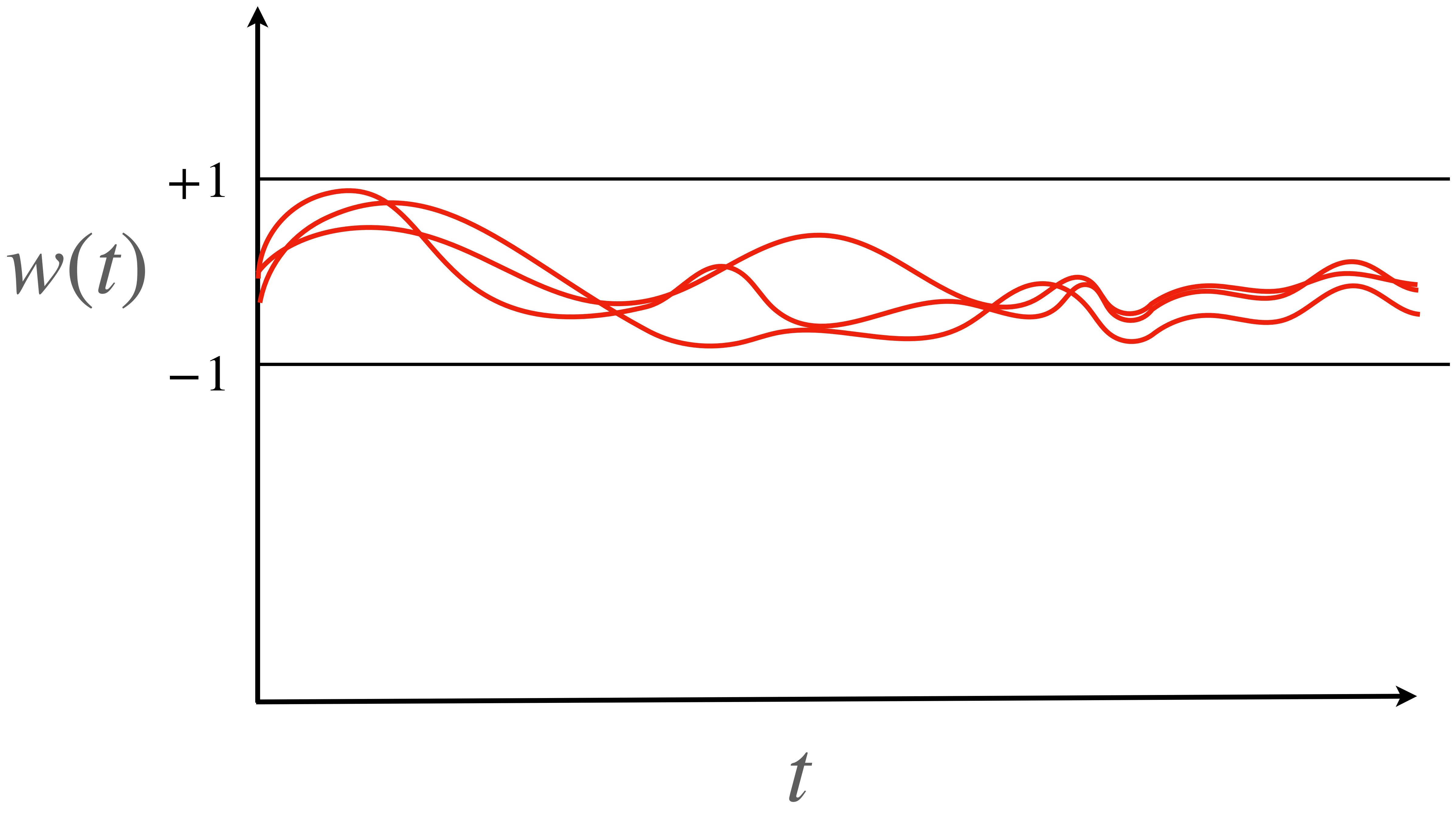


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If $w_1^2 + w_2^2 + w_3^2 \leq 625$ then the set is