Oscillation

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Recap: Fixed points and stability

How to calculate equilibrium/fixed points for discrete time dynamics

There could be multiple fixed points

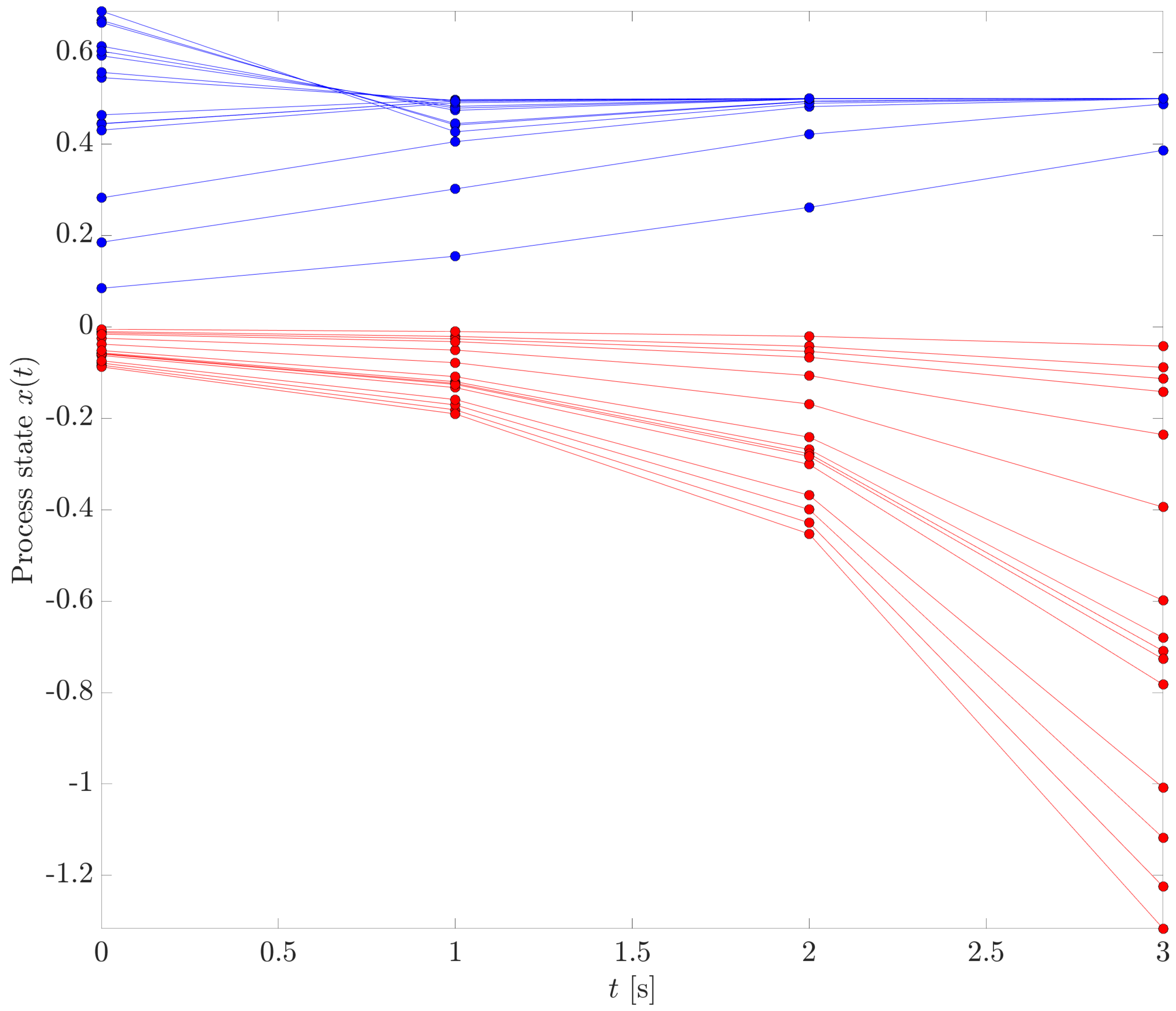
We can simulate the dynamics to investigate if a fixed point is unstable, S, AS, GAS

Recap: Fixed points and stability example

$$x(t+1) = 2x(t)\big(1 - x(t)\big)$$

Two fixed points: $x = 0$, and 0.5

$\rightsquigarrow$ $x = 0$ is unstable, $x = 0.5$ is AS but not GAS



Example: Oscillation in discrete time dynamics

$$x(t+1) = 3.2\, x(t)\big(1 - x(t)\big)$$

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Which one is unstable, S, AS, GAS?

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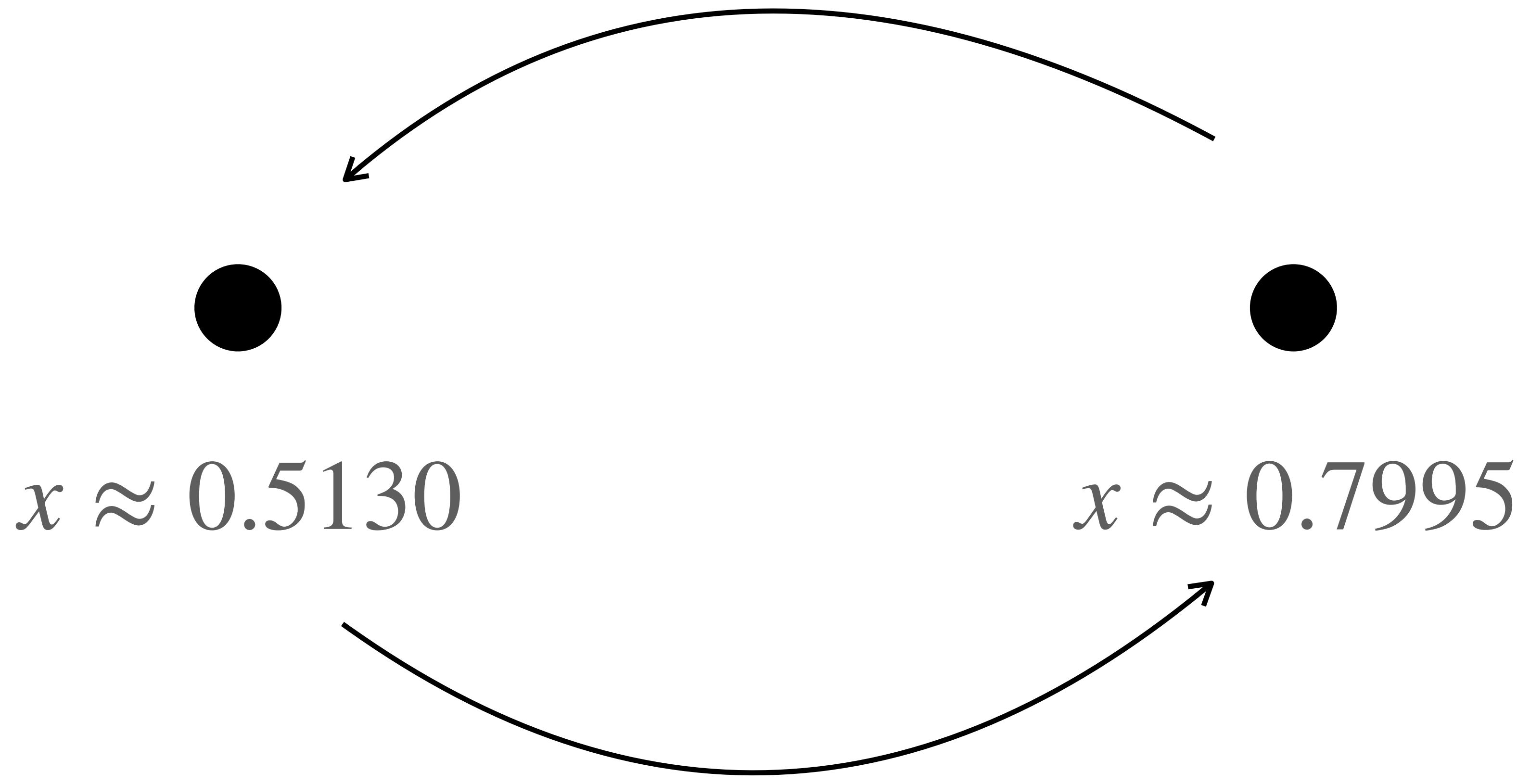
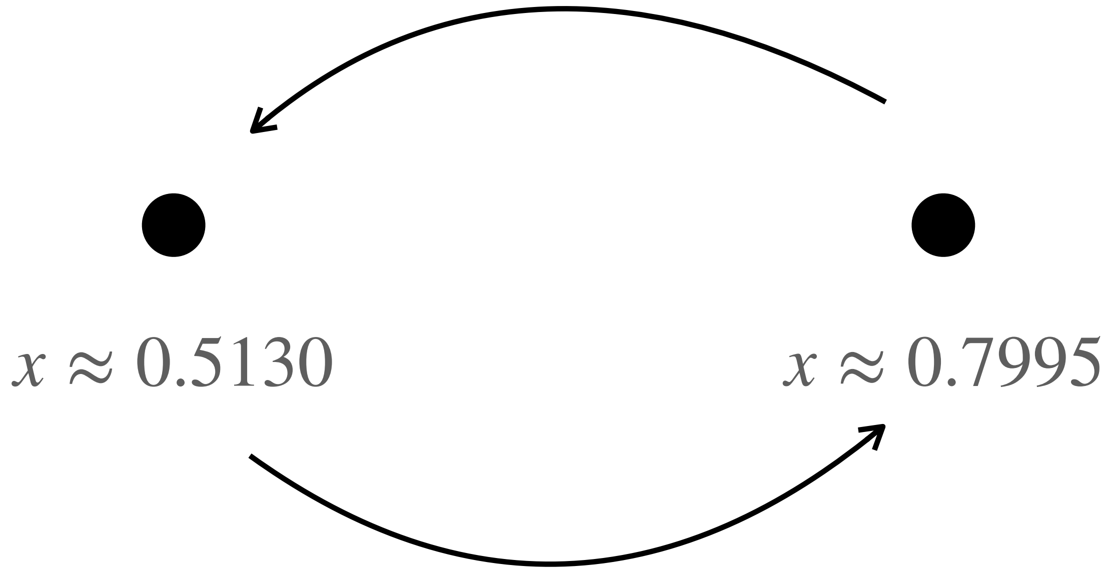
$$x(t+1) = 3.2\,x(t)\big(1 - x(t)\big)$$

Stable oscillation between two points $\Leftrightarrow$ Stable period 2 cycle



Both fixed points $x = 0,\ 0.6875$ are unstable!!

Example: Oscillation in discrete time dynamics



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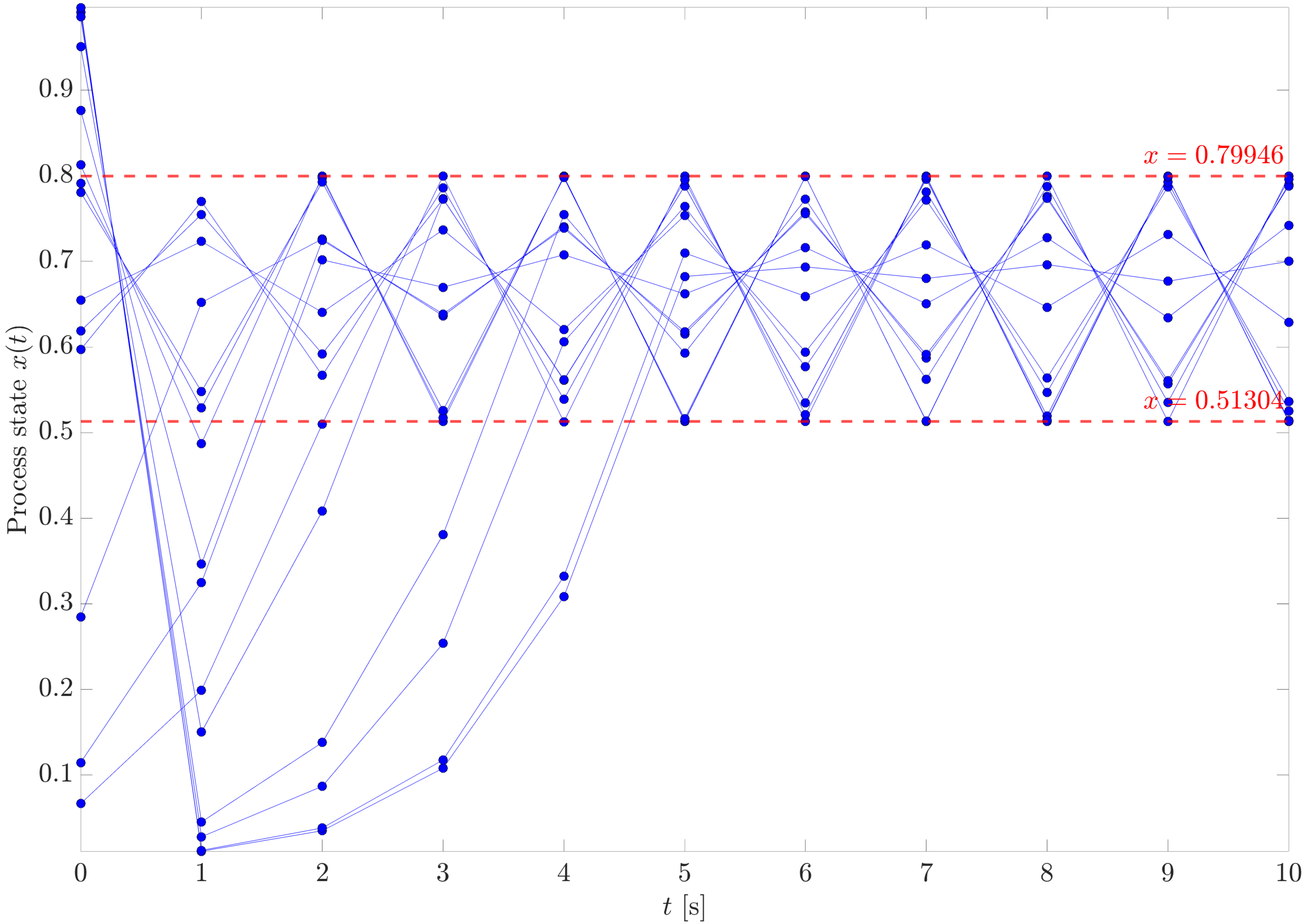

How can we analyze such things?

The discrete time process dynamics is a recursion of the form $x(t+1) = f(x(t))$

The previous example is the specific case $f(x) = rx(1-x), \quad 0 \leq r \leq 4, \quad f : [0,1] \mapsto [0,1]$

Fixed points are the solutions/roots of the equation: $x = f(x)$

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Fixed points are the solutions/roots of the equation: $x = f(x)$

For the specific $f(x) = rx(1-x)$, we get two solutions: $x = 0, \dfrac{r-1}{r}$

So we always have **two** fixed points

How can we analyze such things?

Period 2 points are the solutions/roots of the equation: $x = f(f(x))$

For the specific $f(x) = rx(1-x)$, we get four solutions: $x = 0, \dfrac{r-1}{r}, \dfrac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$

First two are the already known fixed points

Two period 2 solutions for $r > 3$

How can we analyze such things?

Period 2 points are the solutions/roots of the equation: $x = f(f(x))$

For the specific $f(x) = rx(1-x)$, we get:

$$\begin{aligned} x &= rf(x)(1-f(x)) \\ &= r^2x(1-x)\big(1 - rx(1-x)\big) \\ &= r^2x(1-x)\big(1 - rx + rx^2\big) \end{aligned}$$

$$\Rightarrow x\left[1 - r^2(1-x)(1-rx+rx^2)\right] = 0$$

Factor the left hand side as $x\left(x - \dfrac{r-1}{r}\right)$ (another quadratic expression in x)