Pontryagin's Method of Indirect Adjoints for Optimal Control with State Inequality Constraints *A Concrete Example*

Abhishek Halder

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Background

To understand how to apply Pontryagin's Maximum Principle (PMP) in this case, I tried to work out the following example from **A.E. Bryson Jr., and Y.C. Ho, *Applied Optimal Control*, Taylor and Francis, 1975**, for which they list the answer but do not show the calculations. A paper by **A.E. Bryson Jr., W.F. Denham, and S.E. Dreyfus, "Optimal Programming Problems with Inequality Constraints I: Necessary Conditions for Extremal Solutions", *AIAA Journal*, Vol. 1, No. 11, pp. 2544–2550, 1963,** also states this exact problem and lists the answer (Section 6 in the paper), but again sans any calculation that leads to the answer. In both the above references, it was mentioned that the problem was suggested by John V. Breakwell of Lockheed Missiles and Space Company. Other than the two mentioned, I am not aware of any reference where the solution of this problem has been worked out. In both the references, there appears to be a typo in the depiction (Fig. 3.11.3 in the book, and Fig. 3 in the paper) of the solution of this problem: $\ell < \frac{1}{4}$ in those figures would probably be $\ell < \frac{1}{6}$. Nonetheless, my objective here is to understand the details of the PMP calculations involved in solving this problem.

Bryson and Ho (1975), Ch. 3, pg. 120, Example 2

Problem statement

Cost: $\underset{u(t)}{\textbf{minimize}} \quad \frac{1}{2}\int_0^1 u^2\,dt$

Dynamics: $\dot{x}_1 = x_2, \quad \dot{x}_2 = u$

State inequality constraint: $x_1(t) \leq \ell$

Initial condition: $x_1(0) = 0$, $x_2(0) = 1$

Final condition: $x_1(1) = 0$, $x_2(1) = -1$

Solution without the state inequality constraint

Hamiltonian: $H = L + \Lambda^\top f = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$

FOOC: $\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \Rightarrow \lambda_1(t) = c_1, \qquad \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \Rightarrow \lambda_2(t) = -c_1 t + c_2$

PMP: $\frac{\partial H}{\partial u} = u + \lambda_2 = 0 \Rightarrow u = -\lambda_2$

Transversality: $-\lambda_1(T)\underbrace{dx_1(T)}_{=0} - \lambda_2(T)\underbrace{dx_2(T)}_{=0} + H(T)\underbrace{dT}_{=0} = 0 \Rightarrow$ **no new information**

Optimal synthesis:

We have $x_2(t) = \int u(t)\,dt = -\int \lambda_2(t)\,dt = \int (c_1 t - c_2)\,dt \Rightarrow x_2(t) = c_1\frac{t^2}{2} - c_2 t + k_2$. Similarly, $x_1(t) = \int x_2(t)\,dt = c_1\frac{t^3}{6} - c_2\frac{t^2}{2} + k_2 t + k_1$. From initial conditions, $x_2(0) = 1 \Rightarrow k_2 = 1$, and $x_1(0) = 0 \Rightarrow k_1 = 0$. From final conditions, $x_1(1) = 0 \Rightarrow c_1 - 3c_2 = -6$, and $x_2(1) = -1 \Rightarrow c_1 - 2c_2 = -2$, which solved together yields $c_1 = 0, c_2 = 2$.

Thus, the solution is given by: $x_1^*(t) = t\,(1-t)$, $x_2^*(t) = (1-2t)$, $\lambda_1^*(t) = 0$, $\lambda_2^*(t) = 2$, $u^*(t) = -2$.

By direct differentiation, notice that $(x_1^*(t))_{\max} = \frac{1}{4}$ occurs at $t_1 = \frac{1}{2}$.

Solution with the state inequality constraint

Adjoint:

We will take time derivative of the constraint surface $S \leq 0$ until u appears explicitly. $S = x_1 - \ell \leq 0$, $\dot{S} = \dot{x}_1 = x_2$ (still u does not appear explicitly), $\ddot{S} = \dot{x}_2 = u$ (now it does). So we will use $\ddot{S}$ as the adjoint.

Adjoint costate: $\mu(t) = \begin{cases} \geq 0 & \textbf{for} \quad S = 0, \\ = 0 & \textbf{for} \quad S < 0. \end{cases}$

Hamiltonian: $H = L + \Lambda^\top f + \mu(t)\ddot{S} = \dfrac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u + \mu u$

FOOC: $\dot{\lambda}_1 = -\dfrac{\partial H}{\partial x_1} = 0 \Rightarrow \lambda_1(t) = c_1, \quad \dot{\lambda}_2 = -\dfrac{\partial H}{\partial x_2} = -\lambda_1 \Rightarrow \lambda_2(t) = -c_1 t + c_2$

PMP: $\dfrac{\partial H}{\partial u} = 0 \Rightarrow$ **for** $S = 0$**,** $u + \lambda_2 + \mu = 0$**; for** $S < 0$**,** $u + \lambda_2 = 0$**.**

Transversality: $-\lambda_1(T) \underbrace{dx_1(T)}_{=0} - \lambda_2(T) \underbrace{dx_2(T)}_{=0} + H(T) \underbrace{dT}_{=0} = 0 \Rightarrow$ **no new information**

Continuity:

From the dynamics, the state trajectories (and hence the optimal one too) will be continuous. However, the costates and control may not be continuous.

Tangency:

$\underbrace{N}_{2\times 1} \triangleq \begin{pmatrix} S \\ \dot{S} \end{pmatrix}_{t=t_1} = \begin{pmatrix} x_1 - \ell \\ x_2 \end{pmatrix}_{t=t_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where t_1 is the time when the state trajectory is at the constraint surface S, i.e., hitting time. Furthermore, $\ddot{S} = u = 0$ on the constraint boundary S.

Jump conditions:

1. Costate jump condition: $\underbrace{\left(\Lambda\left(t_1^+\right)\right)^\top}_{1\times 2} = \underbrace{\left(\Lambda\left(t_1^-\right)\right)^\top}_{1\times 2} - \underbrace{\nu^\top}_{1\times 2} \underbrace{\nabla_x N\big|_{t=t_1}}_{2\times 2}$.

The column vector $\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$ comprises of *constant* multipliers to be determined. On the other hand, Jacobian of the tangent vector N, evaluated at $t = t_1$ becomes

$$\nabla_x N\big|_{t=t_1} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x_1} & \dfrac{\partial N_1}{\partial x_2} \\ \\ \dfrac{\partial N_2}{\partial x_1} & \dfrac{\partial N_2}{\partial x_2} \end{bmatrix}_{t=t_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, the costate jump condition in our case becomes $\lambda_1\left(t_1^+\right) = \lambda_1\left(t_1^-\right) - \nu_1$, and $\lambda_2\left(t_1^+\right) = \lambda_2\left(t_1^-\right) - \nu_2$.

2. Hamiltonian jump condition: $H\left(t_1^+\right) = H\left(t_1^-\right) + \underbrace{\nu^\top}_{1\times 2} \underbrace{\dfrac{\partial N}{\partial t_1}}_{2\times 1}$.

In our case, $\dfrac{\partial N}{\partial t_1} = 0$, and hence the Hamiltonian jump condition yields

$$\frac{1}{2}\left(u^2\left(t_1^+\right) - u^2\left(t_1^-\right)\right) + \underbrace{x_2\left(t_1\right)}_{\text{continuity}} \underbrace{\left(\lambda_1\left(t_1^+\right) - \lambda_1\left(t_1^-\right)\right)}_{=-\nu_1} + \left(\lambda_2\left(t_1^+\right)u\left(t_1^+\right) - \lambda_2\left(t_1^-\right)u\left(t_1^-\right)\right) \\ + \left(\mu\left(t_1^+\right)u\left(t_1^+\right) - \mu\left(t_1^-\right)u\left(t_1^-\right)\right) = 0.$$

Furthermore, from tangency $x_2\left(t_1\right) = 0$, and hence we get

$$\frac{1}{2}\left(u^2\left(t_1^+\right) - u^2\left(t_1^-\right)\right) + \left(\lambda_2\left(t_1^+\right)u\left(t_1^+\right) - \lambda_2\left(t_1^-\right)u\left(t_1^-\right)\right) + \left(\mu\left(t_1^+\right)u\left(t_1^+\right) - \mu\left(t_1^-\right)u\left(t_1^-\right)\right) = 0. \tag{1}$$

Optimal synthesis:

From the solution of the optimal control problem without state inequality constraint, it is clear that if $\ell > \dfrac{1}{4}$, then the optimal state trajectory won't hit the constraint surface S and hence that optimal solution carries through. Thus, we only need to recompute the solution for $\ell \leq \dfrac{1}{4}$. For that, our first job is to determine t_1 by solving the Hamiltonian jump equation derived above. There are two cases to investigate.

Case I. touch and leave:

The adjoint costate $\mu\left(t_1^+\right) = \mu\left(t_1^-\right) = 0$. From PMP, $u\left(t_1^+\right) = -\lambda_2\left(t_1^+\right)$, $u\left(t_1^-\right) = -\lambda_2\left(t_1^-\right)$, and hence the Hamiltonian jump equation (1) yields

$$\lambda_2^2\left(t_1^+\right) = \lambda_2^2\left(t_1^-\right) \Rightarrow \begin{cases} \text{either} \quad \lambda_2\left(t_1^+\right) = \lambda_2\left(t_1^-\right) \Rightarrow \nu_2 = 0, \\ \text{or} \quad \lambda_2\left(t_1^+\right) = -\lambda_2\left(t_1^-\right) \Rightarrow \lambda_2\left(t_1^\pm\right) = \mp\dfrac{\nu_2}{2}. \end{cases}$$

Case I.A. $\lambda_2\left(t_1^+\right) = \lambda_2\left(t_1^-\right) \Rightarrow \nu_2 = 0$

Consider two time intervals $\left[0, t_1^-\right]$ and $\left[t_1^+, 1\right]$. For $t \in \left[0, t_1^-\right]$, let $\lambda_1(t) = c_1$ and $\lambda_2(t) = -c_1 t + c_2$. For $t \in \left[t_1^+, 1\right]$, let $\lambda_1(t) = \widetilde{c}_1$ and $\lambda_2(t) = -\widetilde{c}_1 t + \widetilde{c}_2$.

For the interval $\left[0, t_1^-\right]$, we have the boundary conditions $x_1(0) = 0, x_2(0) = 1$, and $x_1\left(t_1^-\right) = x_1(t_1) = \ell$, $x_2\left(t_1^-\right) = x_2\left(t_1\right) = 0$ (using continuity and tangency). Similarly, for the interval $\left[t_1^+, 1\right]$, we have the boundary conditions $x_1\left(t_1^+\right) = x_1(t_1) = \ell, x_2\left(t_1^+\right) = x_2\left(t_1\right) = 0$ (using continuity and tangency), and $x_1\left(1\right) = 0$, $x_2\left(1\right) = -1$. Enforcing these boundary conditions yield four equations:

$$c_1\frac{t_1^2}{2} - c_2 t_1 + 1 = \underbrace{x_2\left(t_1\right)}_{=0} = \widetilde{c}_1\frac{t_1^2}{2} - \widetilde{c}_2 t_1 + \left(\widetilde{c}_2 - \frac{\widetilde{c}_1}{2} - 1\right), \tag{2}$$

$$c_1\frac{t_1^3}{6} - c_2\frac{t_1^2}{2} + t_1 = \underbrace{x_1\left(t_1\right)}_{=\ell} = \widetilde{c}_1\frac{t_1^3}{6} - \widetilde{c}_2\frac{t_1^2}{2} + \left(\widetilde{c}_2 - \frac{\widetilde{c}_1}{2} - 1\right)t_1 + \left(\frac{\widetilde{c}_1}{3} - \frac{\widetilde{c}_2}{2} + 1\right). \tag{3}$$

Enforcing (jump condition) $\lambda_2\left(t_1^+\right) = \lambda_2\left(t_1^-\right)$ yields

$$c_1 t_1 - c_2 = \widetilde{c}_1 t_1 - \widetilde{c}_2. \tag{4}$$

We want to solve the above five equations in five unknowns $\left(c_1, c_2, \widetilde{c}_1, \widetilde{c}_2, t_1\right)^\top \in \mathbb{R}^4 \times [0, 1]$.

The above system of five polynomial equations can be solved as

$$\ell \in \mathbb{R} \quad \begin{cases} c_1 = 24\left(1-4\ell\right), \quad \widetilde{c}_1 = -24\left(1-4\ell\right), \quad c_2 = 8\left(1-3\ell\right), \\ \widetilde{c}_2 = 8\left(1-3\ell\right) - 24\left(1-4\ell\right) = -16+72\ell, \quad t_1 = \dfrac{1}{2}, \end{cases} \tag{5}$$

$$0 \le t_1 \le 1 \ \Rightarrow \ 0 \le \ell \le \frac{1}{12} \quad \begin{cases} c_1 =, \quad \widetilde{c}_1 =, \quad c_2 =, \\ \widetilde{c}_2 =, \quad t_1 = \dfrac{1 \pm \sqrt{1-12\ell}}{2}, \end{cases} \tag{6}$$

Case I.B. $\lambda_2\left(t_1^+\right) = -\lambda_2\left(t_1^-\right) \Rightarrow \nu_2 = 0$

The first four equations remain unchanged. The fifth equation is modified to

$$c_1 t_1 - c_2 = -\widetilde{c}_1 t_1 + \widetilde{c}_2. \tag{7}$$

The resulting set of five equations can be solved for t_1 as

$$t_1 = \frac{1 \pm \sqrt{\dfrac{1-6\ell}{1+6\ell}}}{2}. \tag{8}$$

Now, $t_1 \in [0,1]$ requires

$$0 \le \frac{1 \pm \sqrt{\dfrac{1-6\ell}{1+6\ell}}}{2} \le 1 \ \Rightarrow \ -1 \le \pm\sqrt{\frac{1-6\ell}{1+6\ell}} \le 1 \ \Rightarrow \ 0 \le \frac{1-6\ell}{1+6\ell} \le 1.$$

First, consider the *lower bound*, i.e., $0 \le \dfrac{1-6\ell}{1+6\ell}$, which means both the numerataor and the denominator have the same sign. This yields either $\ell \in (-\frac{1}{6}, \frac{1}{6}]$ or $\ell \in [\frac{1}{6}, -\frac{1}{6})$, clearly the latter being impossible. In other words, the numerator must be ≥ 0, and denominator must be > 0. This permits us to multiply the *upper bound* of the above inequality by $(1+6\ell)$, resulting

$$1-6\ell \le 1+6\ell \ \Rightarrow \ 0 \le \ell,$$

which together with the lower bound condition $\ell \in (-\frac{1}{6}, \frac{1}{6}]$, results in $0 \le \ell \le \frac{1}{6}$. Thus, the solution for $(c_1, c_2, \widetilde{c}_1, \widetilde{c}_2, t_1)^\top \in \mathbb{R}^4 \times [0,1]$ in this case, is

$$0 \le t_1 \le 1 \ \Rightarrow \ 0 \le \ell \le \frac{1}{6} \quad \begin{cases} c_1 =, \quad \widetilde{c}_1 =, \quad c_2 =, \\ \widetilde{c}_2 =, \quad t_1 = \dfrac{1 \pm \sqrt{\dfrac{1-6\ell}{1+6\ell}}}{2}, \end{cases} \tag{9}$$

Case II. touch, then hold, then leave:

Let t_1 be the time when it touches the boundary for the first time (hitting/entry time), then holds the boundary S for a finite amount of time, and then leaves S at time t_2 (exit time). So the duration that the trajectory spends on the boundary is $t_2 - t_1$ (sojourn time).

The adjoint costates $\mu\left(t_1^-\right)=0$, $\mu\left(t_1^+\right)\geq 0$, $\mu\left(t_2^-\right)\geq 0$, $\mu\left(t_2^+\right)=0$. From PMP, $u\left(t_1^-\right)=-\lambda_2\left(t_1^-\right)$, and $u\left(t_1^+\right)=-\lambda_2\left(t_1^+\right)-\mu\left(t_1^+\right)$. Similarly, $u\left(t_2^+\right)=-\lambda_2\left(t_2^+\right)$, and $u\left(t_2^-\right)=-\lambda_2\left(t_2^-\right)-\mu\left(t_2^-\right)$. We use these adjoint costate relations and PMP to specialize (1) at entry time t_1, resulting

$$\frac{1}{2}\left(u^2\left(t_1^+\right)-\lambda_2^2\left(t_1^-\right)\right)\;+\;\left(\lambda_2\left(t_1^+\right)u\left(t_1^+\right)+\lambda_2^2\left(t_1^-\right)\right)\;+\;\left(\underbrace{\mu\left(t_1^+\right)u\left(t_1^+\right)}_{\overset{\text{(PMP)}}{=}-\lambda_2(t_1^+)u(t_1^+)-u^2(t_1^+)}-0\right)=0 \tag{10}$$

$$\Rightarrow\quad u^2\left(t_1^+\right)=\lambda_2^2\left(t_1^-\right)\quad\Rightarrow\quad u\left(t_1^+\right)=\pm\lambda_2\left(t_1^-\right). \tag{11}$$

Likewise, specializing (1) at exit time t_2, results

$$\frac{1}{2}\left(\lambda_2^2\left(t_2^+\right)-u^2\left(t_2^-\right)\right)+\left(-\lambda_2^2\left(t_2^+\right)-\lambda_2\left(t_2^-\right)u\left(t_2^-\right)\right)+\left(0-\underbrace{\mu\left(t_2^-\right)u\left(t_2^-\right)}_{\overset{\text{(PMP)}}{=}-\lambda_2(t_2^-)u(t_2^-)-u^2(t_2^-)}\right)=0 \tag{12}$$

$$\Rightarrow\qquad u^2\left(t_2^-\right)=\lambda_2^2\left(t_2^+\right)\quad\Rightarrow\quad u\left(t_2^-\right)=\pm\lambda_2\left(t_2^+\right). \tag{13}$$

Now we use the tangency condition that $\ddot{S}=u=0$ on the constraint boundary S. This gives $u(t)=0\,\forall t\in\left[t_1^+,t_2^-\right]$, and in particular, $u\left(t_1^+\right)=u\left(t_2^-\right)=0\Rightarrow\lambda_2\left(t_1^-\right)=\lambda_2\left(t_2^+\right)=0$. Invoking the continuity of the states x_1 and x_2, we get the a pair of polynomial systems. At the entry time t_1, we get

$$\underbrace{x_2\left(t_1\right)}_{=0}\;=\;c_1\frac{t_1^2}{2}-c_2t_1+1, \tag{14}$$

$$\underbrace{x_1\left(t_1\right)}_{=\ell}\;=\;c_1\frac{t_1^3}{6}-c_2\frac{t_1^2}{2}+t_1, \tag{15}$$

$$\underbrace{\lambda_2\left(t_1^-\right)}_{=0}\;=\;-c_1t_1+c_2. \tag{16}$$

At the exit time t_2, we get

$$\underbrace{x_2\left(t_2\right)}_{=0}\;=\;\widetilde{c_1}\frac{t_2^2}{2}-\widetilde{c_2}t_2+\left(\widetilde{c_2}-\frac{\widetilde{c_1}}{2}-1\right), \tag{17}$$

$$\underbrace{x_1\left(t_2\right)}_{=\ell}\;=\;\widetilde{c_1}\frac{t_2^3}{6}-\widetilde{c_2}\frac{t_2^2}{2}+\left(\widetilde{c_2}-\frac{\widetilde{c_1}}{2}-1\right)t_2+\left(\frac{\widetilde{c_1}}{3}-\frac{\widetilde{c_2}}{2}+1\right), \tag{18}$$

$$\underbrace{\lambda_2\left(t_2^+\right)}_{=0}\;=\;-\widetilde{c_1}t_2+\widetilde{c_2}. \tag{19}$$

Solving eqs. (14)–(16) results

$$c_1=\frac{2}{9\ell^2},\quad c_2=\frac{2}{3\ell},\quad t_1=3\ell, \tag{20}$$

and solving eqs. (17)–(19) yields

$$\widetilde{c_1}=-\frac{2}{9\ell^2},\quad \widetilde{c_2}=\frac{2}{9\ell^2}\left(3\ell-1\right),\quad t_2=1-3\ell. \tag{21}$$

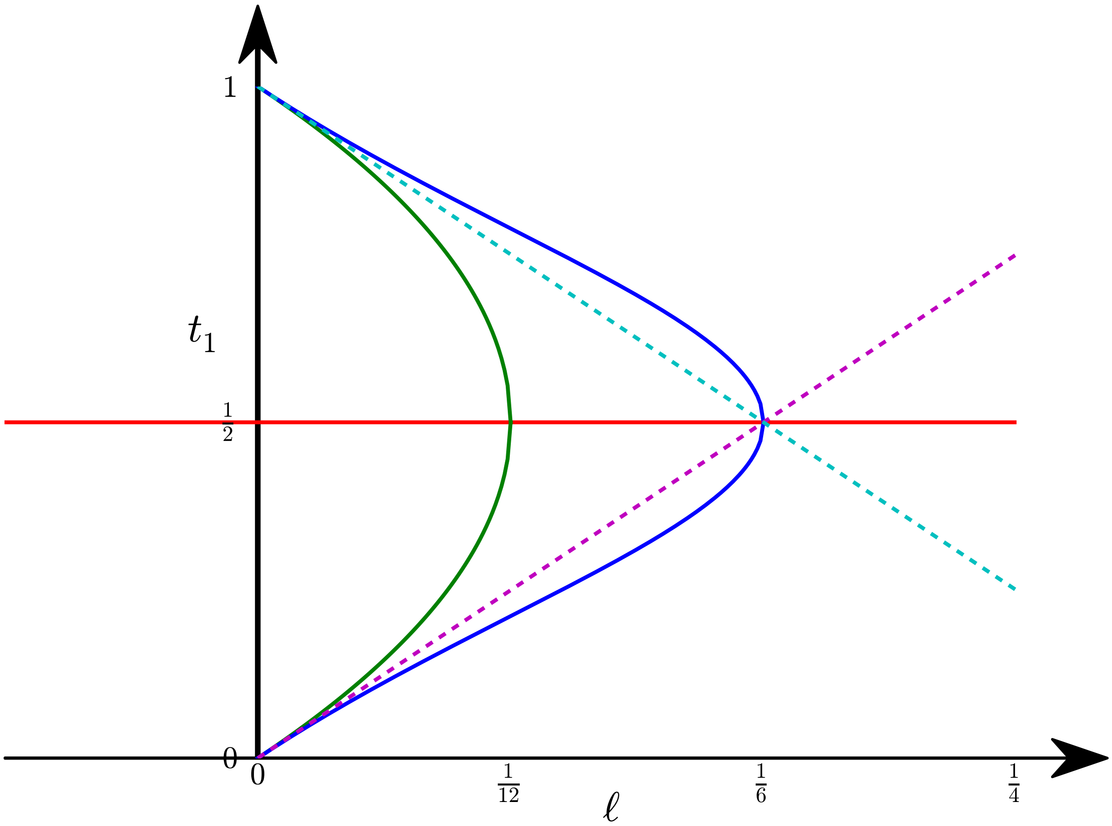

Figure 1: Bifurcation diagram showing the variation of switching times as a function of the parameter $\ell \in (-\infty, \frac{1}{4}]$.

However, for both t_1 and t_2 to be within 0 to 1, we need $0 \leq \ell \leq \frac{1}{3}$, which together with $\ell \leq \dfrac{1}{4}$ results the parametric range $0 \leq \ell \leq \dfrac{1}{4}$. Thus, the solution is

t	$x_1^*(t)$	$x_2^*(t)$	$\lambda_1^*(t)$	$\lambda_2^*(t)$	$u^*(t)$
$0 \leq t \leq 3\ell$	$-\ell\left(1-\dfrac{t}{3\ell}\right)^3$	$x_2^*(t)$	$\lambda_1^*(t)$	$\dfrac{2}{3\ell}\left(1-\dfrac{t}{3\ell}\right)$	$-\dfrac{2}{3\ell}\left(1-\dfrac{t}{3\ell}\right)$
$3\ell \leq t \leq 1-3\ell$	ℓ	0	$\lambda_1^*(t)$	$\lambda_2^*(t)$	0
$1-3\ell \leq t \leq 1$	$x_1^*(t)$	$x_2^*(t)$	$\lambda_1^*(t)$	$\dfrac{2}{3\ell}\left(1-\dfrac{1-t}{3\ell}\right)$	$-\dfrac{2}{3\ell}\left(1-\dfrac{1-t}{3\ell}\right)$

Table 1: Solution for Case II with admissible parametric range $0 \leq \ell \leq \frac{1}{4}$.